Mathematics 600, September - December 2009. Instructor: Reichstein.

Problem set 4. Due Tuesday, December 1

Problem 1: Let $V_0 = \mathbb{C}^n$ with the standard basis $e_1, \dots, e_n$,

$$q_0(x_1 e_1 + \cdots + x_n e_n) = x_1^2 + \cdots + x_n^2$$

be the standard quadratic form on V_0 and

$$\omega_0 = e_1^* \wedge \cdots \wedge e_n^* \in \Lambda^n(V_0)$$

be the standard volume form on V_0.

Let K be a field extension of $\mathbb{C}$. A K-form of $(\mathbb{C}^n, q_0, \omega_0)$ is a triple (V, q, ω), where V is an n-dimensional vector space, equipped with a quadratic form $q\colon V \to K$ and a volume form $\omega \in \Lambda^n(V)$ such over after tensoring with the algebraic closure $\overline{K}$ of K, (V, q, ω) and $(\mathbb{C}^n, q_0, \omega_0)$ become isomorphic. Show that the functors

$$F_1\colon K \mapsto \{\, K\text{-forms of } (\mathbb{C}^n, q_0, \omega_0), \text{ up to } K\text{-isomorphism} \,\}$$

and

$$F_2\colon K \mapsto \{n\text{-dimensional quadratic forms of discriminant 1, up to } K\text{-isomorphism} \,\}$$

are isomorphic. Conclude that $\mathrm{ed}(\mathrm{SO}_n) = n - 1$ for every $n \geq 3$.

Problem 2: Let X be an irreducible faithful S_n-variety, $K = \mathbb{C}(X)^{S_n}$, and A be the n-dimensional $\mathbb{C}$-algebra $\mathbb{C} \times \cdots \times \mathbb{C}$, with componentwise addition and multiplication and the $\mathbb{C}$-algebra structure given

$$c \cdot (a_1, \dots, a_n) = (ca_1, \dots, ca_n)\,.$$

(a) Show that $L = \mathrm{RMaps}_{S_n}(X, A)$ is a field extension of K of degree n.

(b) What is the relationship between L and $\mathbb{C}(X)$? Hint: $[L : K] = n$ and $[\mathbb{C}(X) : K] = |S_n| = n!$.

(c) Conversely, show that every finite field extension L/K of degree n is of the form $\mathrm{RMaps}_{S_n}(X, A)$, where X is a primitive faithful S_n-variety uniquely determined by L/K.

Problem 3: (a) Construct a rational quotient map π for the natural multiplication action of the symplectic group Sp_{2n} on the n^2-dimensional vector space M_n of $n \times n$-matrices.

(b) Construct a rational section for π. Conclude that $\mathrm{ed}(\mathrm{Sp}_{2n}) = 0$.

Problem 4: Let $1 \to N \to G \to G/N \to 1$ be a short exact sequence of complex linear algebraic groups and K be a finitely generated field extension of $\mathbb{C}$. Show that the induced "long" cohomology sequence

$$1 \to N(K) \to G(K) \to (G/N)(K) \to H^1(K, N) \to H^1(K, G) \to H^1(K, G/N)$$

is exact at $H^1(K, N)$.