Problem set 4. Due Thursday, December 1.

Mathematics 600, Term 2, 2016. Instructor: Reichstein.

As usual, $F$ is assumed to be a field of characteristic $\neq 2$.

(1) Let $F$ be a field containing $\sqrt{-1}$ and $q = \langle a_1, \ldots, a_n \rangle$ be an even-dimensional form of discriminant 1. Show that the Clifford invariant of $q$ is given by the formula

$$C(q) = \otimes_{1 \leq i \leq j \leq n} \left( \frac{a_i a_j}{F} \right)$$

in $\text{Br}(F)$.

Remark: Since $-1$ is a square in $F$, there is no difference between the signed discriminant $d_{\pm}(q)$ and and the usual discriminant $d(q)$. Both are equal to $a_1 \ldots a_n \pmod{(F^*)^2}$.

(2) Recall that the Albert form $q_{A,B}$ of two quaternion algebras, $A = \left( \frac{a_1, a_2}{F} \right)$ and $B = \left( \frac{b_1, b_2}{F} \right)$, is the difference of their norm forms. Explicitly, $q_{A,B} = \langle -a_1, -a_2, a_1 a_2, b_1, b_2, -b_1 b_2 \rangle$.

Show that the Clifford algebra $C(q)$ is Brauer equivalent to $A \otimes_F B$ over $F$.

(3) p. 141, Exercise 5.

(4) Let $E/F$ be a Galois field extension, with Galois group $\text{Gal}(E/F) \cong (\mathbb{Z}/2\mathbb{Z})^n$. Show that the trace form $x \rightarrow \text{Tr}_{E/F}(x^2)$ is the scaled Pfister form $\langle 2^n \rangle \otimes_F \langle \langle a_1, \ldots, a_n \rangle \rangle$ for some $a_1, \ldots, a_n \in F^*$.

(5) Let $q = \langle a_1, \ldots, a_n \rangle$. Assume $F$ contains $\sqrt{-1}$.

(a) Show that the (non-reduced) trace form of the Clifford algebra $C(q)$ is the scaled Pfister form $\langle 2^n \rangle \otimes_F \langle \langle a_1, \ldots, a_n \rangle \rangle$.

(b) Conclude that if $\langle a_1, \ldots, a_n \rangle \cong \langle a_1, \ldots, a_n \rangle$ then

$$\langle \langle a_1, \ldots, a_n \rangle \rangle \cong \langle \langle b_1, \ldots, b_n \rangle \rangle.$$

Remark: Recall that the non-reduced trace form of a finite-dimensional associative $F$-algebra $A$ is $a \rightarrow \text{Tr}(l_a^2)$, where $l_b: A \rightarrow A$

is the left multiplication map, $l_b(x) := bx$. 