Problem set 3. Due Thursday, November 17.

Mathematics 600, Term 2, 2016. Instructor: Reichstein.

In Problems 1–4, \( n \) will denote a positive integer, \( F \) will denote a field containing a primitive root of unity \( \zeta \in F \), and \( a, b \) will denote non-zero elements of \( F \). Define the “cyclic algebra” \( A = \left( \frac{a, b}{F} \right)_\zeta \) by generators \( x, y \) and relations \( x^n = a, \ y^n = b \) and \( xy = \zeta yx \). Note that is a direct generalization of the definition of the quaternion algebra \( A = \left( \frac{a, b}{F} \right) \), where \( n = 2 \). Here \( \zeta = -1 \) is the unique primitive \( n \)th root of unity, and we write \( \left( \frac{a, b}{F} \right) \) in place of \( \left( \frac{a, b}{F} \right)_{-1} \).

1. Show that \( \left( \frac{a, b}{F} \right)_\zeta \) is isomorphic to \( \left( \frac{as^n, bt^n}{F} \right)_{s,t} \zeta \) as an \( F \)-algebra, for any \( s, t \in F^* \).

2. Show that \( \left( \frac{1, b}{F} \right)_\zeta \) is isomorphic to the matrix algebra \( M_n(F) \), as an \( F \)-algebra.

3. Show that \( \left( \frac{a, b}{F} \right)_{\zeta} \) is a central simple algebra of degree \( n \).

4. Conversely, suppose \( A \) is a central simple algebra of degree \( n \). Show that \( A = \left( \frac{a, b}{F} \right)_{\zeta} \) for some \( a, b \in F^* \) if and only if there exists an element \( x \in A \) such that \( x^n = a \) but \( x^m \not\in F \) for any \( 1 \leq m \leq n - 1 \). Here, as usual, I am identifying \( F \) with \( F \cdot 1 \subset A \).

5. If \( A = \left( \frac{a, b}{F} \right)_{\zeta} \), show that \( A^{op} \simeq \left( \frac{b, a}{F} \right)_{\zeta} \).

6. Show that \( M_m(F) \otimes_F M_n(F) \) is isomorphic to \( M_{mn}(F) \) as an \( F \)-algebra.
(7) Suppose \( \text{char}(F) \neq 2 \). Show that every central simple \( F \)-algebra of degree 2 is isomorphic to the quaternion algebra \( \left( \frac{a, b}{F} \right) \) for some \( a, b \in F^* \).

(8) Let \( A \) and \( B \) be central simple algebras over \( F \). Denote their trace forms by \( q_A \) and \( q_B \). Assume that \( \text{char}(F) = 0 \) or \( \text{char}(F) > \max\{\deg(A), \deg(B)\} \). Show that the trace form of \( A \otimes_F B \) is \( q_A \otimes q_B \).

Recall that the trace form of \( A \) is the quadratic form \( x \mapsto \text{Tr}_{A/F}(x^2) \).

(9) Let \( F = \mathbb{C}(a_1, a_2, b_1, b_2) \), where \( a_1, a_2, b_1, b_2 \) are independent variables and \( A = \left( \frac{a_1, a_2}{F} \right) \), \( B = \left( \frac{b_1, b_2}{F} \right) \) be quaternion algebras. Show that the trace form of \( A \otimes_F B \) is anisotropic over \( F \).

(10) Let \( F, A \) and \( B \) be as in Problem (9). Show that \( A \otimes_F B \) is not cyclic, i.e., is not \( F \)-isomorphic to a central simple algebra of the form \( C = \left( \frac{c_1, c_2}{F} \right) \zeta \) for any primitive 4th root of unity \( \zeta \in \mathbb{C} \) and any \( c_1, c_2 \in F^* \).

**Hint:** Check that the trace form of \( C \) is isotropic.