Problem set 5. Due Thursday, November 16.
Mathematics 534, Term 1, 2017. Instructor: Reichstein.

All Lie algebras and all vector spaces in the problems below are assumed to be finite-dimensional and defined over the field $\mathbb{C}$ of the complex numbers.

(1) Let $L$ be a (finite-dimensional) Lie algebra and $V_1, \ldots, V_m, W_1, \ldots, W_n$ be irreducible $L$-modules. If $V_1 \oplus \cdots \oplus V_m$ is isomorphic to $W_1 \oplus \cdots \oplus W_n$, as an $L$-module, then $m = n$ and there exists a permutation $\sigma \in S_n$ such that $W_i \simeq V_{\sigma(i)}$ for every $i = 1, \ldots, n$.

Remark: In the case, where $L$ is semisimple, this assertion may be viewed as the “uniqueness part” of Weyl’s theorem.

(2) Page 40, problem 5.

(3) Recall that $\mathfrak{so}_n$ is the Lie algebra of skew-symmetric $n \times n$ matrices.
   (a) Show that $\mathfrak{so}_2$ is toral.
   (b) Show that $\mathfrak{so}_3$ is isomorphic to $\mathfrak{sl}_2$.
   (c) Let $H$ be the the image of $\mathfrak{so}_2 \times \cdots \times \mathfrak{so}_2$, diagonally embedded into $\mathfrak{so}_{2n}$. Show that $H$ is a maximal toral subalgebra of $\mathfrak{so}_n$.

(4) (a) Find the roots of $\mathfrak{so}_4$ relative to the maximal toral subalgebra $H \simeq \mathfrak{so}_2 \times \mathfrak{so}_2$ constructed in the previous problem. For each root $\alpha: H \to \mathbb{C}$ of $\mathfrak{so}_4$ find the associated root space $L_\alpha$. That is, find a matrix $x_\alpha \in \mathfrak{so}_4$ such that $L_\alpha = \text{Span}_\mathbb{C}(x_\alpha)$.
   (b) Do the same for $\mathfrak{so}_{2n}$, for every $n \geq 2$.

(5) p. 41, Problem 8.

(6) If $H$ is semisimple Lie algebra and $H$ is a maximal toral subalgebra, show that $\dim(L) \geq 3 \dim(H)$. Use this inequality to solve Problem 10 on p. 41.