Problem set 4. Due Thursday, November 2.
Mathematics 534, Term 1, 2017. Instructor: Reichstein.

All Lie algebras and all vector spaces in the problems below are assumed to be finite-dimensional and defined over the field $\mathbb{C}$ of the complex numbers.

(1) Let $L_1$ and $L_2$ be Lie algebras and $\phi: L_2 \to \text{Der}(L_1)$ be a homomorphism. Here $\text{Der}(L_1)$ denotes the algebra of derivations $L_1 \to L_1$. Recall that we defined the semidirect product $L_1 \rtimes_\phi L_2$ as the vector space $L_1 \oplus L_2$ with the Lie bracket

$$[(a_1, a_2), (b_1, b_2)] = ([a_1, b_1] + \phi(a_2)(b_1) - \phi(b_2)(a_1), [a_2, b_2]).$$

Here $a_1, b_1 \in L_1$ and $a_2, b_2 \in L_2$. Show that $L_1 \rtimes_\phi L_2$ satisfies the axioms of a Lie algebra.

Hint: Use the fact that the Jacobi identity (L3) on p. 1 in the book is trilinear in $x, y$ and $z$ to simplify your computations.

(2) Let $L$ be a Lie algebra and $R$ be the radical of $L$. Show that $[L, R] = [L, L] \cap R$.

(3) Let $L$ be a Lie algebra of dimension $\geq 2$. Assume that $L$ is not simple. Show that there exists an ideal $(0) \subsetneq I \subsetneq L$ and a subalgebra $S \subset L$ such that $L = I \rtimes S$.

(4) Let $h: L_1 \to L_2$ be a homomorphism of semisimple Lie algebras. If $a = a_s + a_n$ is the Jordan decomposition of $a \in L_1$, show that $h(a) = h(a_s) + h(a_n)$ is the Jordan decomposition of $h(a) \in L_2$.

Hint: The Corollary on p. 30 may be helpful here.

(5) Let $V$ be an $\text{sl}_2$-module. Show that $V$ is self-dual. That is, show that $V$ and $V^*$ are isomorphic as $\text{sl}_2$-modules.

(6) Let $V(m)$ be the unique irreducible $\text{sl}_2$-module of dimension $m + 1$. Decompose $V(m) \otimes V(n)$ as a direct sum of irreducible $\text{sl}_2$-modules.