Problem 4: Let $L$ be a finite-dimensional Lie algebra and $S \subseteq L$ be a solvable subalgebra of dimension $d$ such that $S \neq L$. Show that $L$ has another subalgebra $S'$ such that $S \subseteq S'$ and $\dim(S') = d + 1$. (Note that $S'$ may not be solvable.)

Solution: Consider the representation $\phi: S \to \text{gl}(V)$, where $V = L/S$ given by $\phi(s)(a + S) = \text{ad}(s)(a) + S$ for any $s \in S$ and $a \in L$. This representation is well-defined because $\text{ad}(s)$ preserves $S$.

Since $S$ is solvable, Lie’s Theorem tells us that there exists a non-zero vector $v = a + S \in V$ such that $\text{ad}(s) \cdot v$ is a scalar multiple of $v$ for every $s \in S$. In other words, $\text{ad}(s)a = \lambda(s)a + S$ for some some character $\lambda: S \to F$. Note that in order to apply Lie’s theorem, we need to assume that $F$ is algebraically closed of characteristic 0.

Now set $S' := \text{Span}_F(S,a)$. We claim that
(a) $\dim_F(S') = \dim_F(S) + 1$,
(b) $S$ is a Lie subalgebra of $L$,
(c) $I := \text{Span}(a)$ is a 1-dimensional ideal in $L$,
(d) $S'$ is solvable.

Proof of (a): Since $v \neq 0$, $a \not\in S$, and part (a) follows.

Proof of (b): To see that $S'$ is closed under Lie bracket, take two typical elements, $s'_1 = s_1 + t_1a$ and $s'_2 = s_2 + t_2a$, where $s_1, s_2 \in S$ and $t_1, t_2 \in F$. We want to show that $[s'_1, s'_2]$ lies in $S'$.

$$[s'_1, s'_2] = [s_1, s_2] + [s_3, a],$$

where $s_3 = t_2s_1 - t_1s_2 \in S$. The first term is in $S$ because $S$ is a subalgebra of $L$. The second term equals $\lambda(s_3)a$ and hence, lies in $S'$.

Proof of (c): It suffices to show that $\text{ad}(s')a \in I$ for any $s' \in S'$. Again, write $s' = s + ta$, as in part (b). Then

$$\text{ad}(s')a = \text{ad}(s)a + t \text{ad}(a)a = \lambda(s)a + 0 \in I.$$

Proof of (d): $I$ is commutative and hence, solvable. $S'/I \simeq S$ is solvable by assumption. Hence, $S'$ is solvable. □