Math. 534. Solution to Problem 6, Problem Set 1.

**Problem 6:** Show that the sum $I + J$ of nilpotent ideals $I$ and $I$ in a Lie algebra $L$ is again nilpotent. Here $L$ is assumed to be finite-dimensional but not necessarily nilpotent.

I will present two solutions. Solution 1 relies only on the definition of nilpotency; it works for any Lie algebra (not necessarily finite-dimensional). Solution 2 relies on Engel’s theorem.

**Solution 1.** Recall that $I^n$ is defined recursively as follows: $I^0 = I$, $I^{n+1} = [I, I^n]$. By definition, $I$ is nilpotent if $I^n = (0)$ for some $n$. For convenience, let us set $I^{-n} = L$ for any integer $n < 0$.

I claim that

\[(1) \quad (I + J)^{n+m} \subset I^m + J^n \quad \text{for any integers } m \text{ and } n.\]

If (1) is proved, then the assertion of the problem is immediate: choosing $m$ and $n$ so that $I^m = J^n = (0)$, we obtain $(I + J)^{n+m} = 0$, as desired.

To prove the claim, we argue by induction on $m + n$. The base case, where $n + m < 0$ is obvious: here either $n < 0$ or $m < 0$, so the right hand side of (1) is all of $L$.

For the induction step, we need to prove that $[a + b, x] \in I^m + J^n$ for every $a \in I$, $b \in J$ and $x \in (I + J)^{m+n-1}$. It suffices to show that both $[a, x]$ and $[b, x]$ lie in $I^m + J^n$.

To prove that $[a, x] \in I^m + J^n$, note that by the induction assumption $x \in I^{m+n-1} \subset I^{m-1} + J^n$. Hence,

\[[a, x] \in [a, I^{m-1}] + [a, J^n] \subset I^m + J^n.\]

Indeed, $[a, I^{m-1}] \subset I^m$ by the definition of $I^m$ and $[a, J^n] \subset J^n$ because $J^n$ is an ideal of $L$. Similarly, by the induction assumption $x \in I^{m+n-1} \subset I^m + J^{n-1}$

\[[b, x] \subset [b, I^m] + [b, J^{n-1}] \subset I^m + J^n,\]

as desired. \qed

**Solution 2.** Step 1. There exists a sequence of ideals of $L$,

\[(0) = L_0 \subset L_1 \subset \cdots \subset L_{n-1} \subset L_r = L,\]

such that there are no ideals of $L$ strictly between $L_{i-1}$ and $L_i$. Equivalently, each $L_i/L_{i-1}$ is irreducible as an $L$-module.

To prove this, choose $L_1$ to be a non-trivial ideal of $L$ of minimal dimension, $L_2$ to be the preimage of a non-trivial ideal of $L/L_1$ of
minimal dimension, $L_3$ to be the preimage of a non-trivial ideal of $L/L_2$ of minimal dimension, etc.

Step 2. If $a \in I$, then $\text{ad}(a)(L_i) \subset L_{i-1}$ for each $i = 1, \ldots, r$.

Proof: Set $V_i = L_i/L_{i-1}$ and $W_i = \{ v \in V_i \mid \text{ad}(a)(v) = 0 \ \forall \ a \in I \}$. Since $I$ is an ideal of $L$, $W_i$ is an $\text{ad}(L)$-invariant subspace of $V_i$ (check!), and $W_i \neq (0)$ by Engel’s theorem. Irreducibility of $V_i$ now tells us that $W_i = V_i$. In other words, $\text{ad}(a)(v) = 0$ for any $a \in I$ and any $v \in V_i$. Equivalently, $\text{ad}(a)(x) \in L_{i-1}$ for any $x \in L_i$.

Step 3. By Step 2, $\text{ad}(a)(L_i) \subset L_{i-1}$ for every $a \in I$ and every $i = 1, \ldots, r$. Similarly $\text{ad}(b)(L_i) \subset L_{i-1}$ for $b \in J$. Consequently, $\text{ad}(a + b)(L_i) \subset L_{i-1}$ for every $a \in I$, $b \in J$ and $i = 1, \ldots, r$. We conclude that $\text{ad}(a + b)^r = 0$. Thus $I + J$ is nilpotent by Engel’s Theorem. $\square$