Math 534, Fall 2015. Solutions to Problem Set 3

**Problem 1.** (5pts)
Let \( N \) be a nilpotent ideal of \( L \) of maximal dimension. If \( I \) is another nilpotent ideal, then \( N \) is contained in \( N + I \), which is also a nilpotent ideal. By maximality of \( \dim(N) \), we have \( N + I = N \) and thus \( I \subset N \). This proves both maximality and uniqueness of \( N \).

**Problem 2 (1 on page 20).** (5pts)
Let \( g = \mathfrak{sl}(V) \) and \( R \) be its radical. By Lie’s theorem, there is a \( 0 \neq v \in V \) which is a common eigenvector for every element of \( R \).

Now suppose \( g \in \text{GL}(V) \) be an invertible linear map \( V \to V \). Then conjugation by \( g \), gives rise to a Lie algebra automorphism \( c_g : \mathfrak{sl}(V) \to \mathfrak{sl}(V) \). (Check!) That is, \( c_g(a) := gag^{-1} \) for every \( a \in \mathfrak{sl}(V) \).

In particular, we have \( c_g(R) = R \). Hence, \( g(v) \) is also a common eigenvector for every element of \( R \) (check!). Since every non-zero \( w \in V \) can be written as \( w = g(v) \), for a suitable \( g \in \text{GL}(V) \), we conclude that every \( 0 \neq w \in V \) is a common eigenvector for every element of \( R \).

We claim that every \( a \in R \) is a scalar matrix. Let \( e_1, \ldots, e_n \) be a basis of \( V \). Since every non-zero element of \( V \) is an eigenvector for \( a \), \( a(e_i) = \lambda_i e_i \) for \( i = 1, \ldots, n \). Moreover, since \( e_i + e_j \) is also an eigenvector, we conclude that \( \lambda_i = \lambda_j \). This proves the claim.

**Problem 3 (5 on page 21).** (10pts)
Assume \( x, y \in g \) commute. Then \( x_s \) commutes with \( y \) by proposition 4.2b. Since \( y_s, y_n \) are polynomials of \( y \), \( x_s \) commutes with \( y_s, y_n \). Similarly \( x_s, x_n, y_s, y_n \) all commute with each other.

Since \( x_s, y_s \) commute, they are diagonalized simultaneously, hence \( x_s + y_s \) is semisimple. \( x_n, y_n \) are nilpotent and commuting, hence \( x_n^k = 0, y_n^k = 0 \) when \( k, l \) large enough. Then \( (x_n + y_n)^{k+l} = 0 \) and \( x_n + y_n \) is nilpotent.

Now \( x_s + y_s \) semisimple and \( x_n + y_n \) nilpotent and \( x_s + y_s \) commutes with \( x_n + y_n \). By uniqueness of Jordan decomposition we have \( (x + y)_s = x_s + y_s, (x + y)_n = x_n + y_n \).

Example: Let \( x = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} \) and \( y = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). Here \( x \) is semisimple and \( y \) is nilpotent, so \( x_s = x, x_n = 0, y_s = 0, y_n = y \). However, they don’t commute: \( xy = y, but yx = 2y \).

Note that \( \begin{pmatrix} 1 & 1 \\ 0 & 2 \end{pmatrix} = x + y \), \( x \) is semisimple and \( y \) is nilpotent, but this is not a Jordan decomposition. Since \( x + y \) has 2 different eigenvalues, \( x + y \)
is semisimple and
\[(x + y)s = x + y \neq x_s + y_s, \quad (x + y)_n = 0 \neq x_n + y_n.\]

**Problem 4.** (10pts)
Consider the adjoint representation
\[
\overline{\text{ad}} : S \to \mathfrak{gl}(L/S)
\]
\[s \mapsto \{x \mod S \mapsto [s, x] \mod S\}\]

The image \(\overline{\text{ad}}(S)\) is solvable subalgebra of \(\mathfrak{gl}(L/S)\). Since \(S\) is proper, \(L/S \neq 0\). By Theorem 4.1, \(V = L/S\) has a common eigenvector of \(\overline{\text{ad}}(S)\). In other words, there exists a \(0 \neq \overline{v} \in L/S\) such that \(\overline{\text{ad}}s(\overline{v}) \in \mathbb{C}\overline{v}\). Since \(\overline{v} \neq 0\) in \(L/S\), we can choose a representative \(v \notin S\). Then \(S' := S + \mathbb{C}v\) is a subalgebra of \(L\) (check!), and \(\dim(S') = \dim(S) + 1\).

\(S'\) may not be solvable. Indeed, if it always were, we would be able to use this argument recursively, starting with \(S = (0)\), to show that \(L\) itself is solvable. Equivalently, if \(S\) is a solvable subalgebra of \(L\) of maximal dimension, then \(S'\) cannot be solvable.

**Problem 5.**
(a) (5pts) By Lie’s theorem, there exists a basis of \(\mathbb{C}^n\) such that under this basis \(L\) is contained in the algebra \(t_n\) of upper triangular matrices. Thus \(\dim(L) \leq \dim(t_n) = n(n + 1)/2\).

(b) (5pts) If two solvable subalgebras are of dimension \(n(n + 1)/2\), By Lie’s theorem, \(L_1\) is conjugate to a subalgebra of \(t_n\). Since \(\dim(L_1) = n(n + 1)/2 = \dim(t_n)\), \(L_1\) is conjugate to \(t_n\). By the same argument \(L_2\) is conjugate to \(t_n\). Hence \(L_1\) and \(L_2\) are conjugate to each other.

(c) (5pts) Choose \(n \geq 2\), and let \(L_1 = \{X \in t_n \mid \text{Tr}(X) = 0\}\), and \(L_2 = \{X \in t_n : X_{11} = 0\}\). Then \(\dim(L_1) = \dim(L_2) = n(n + 1)/2 - 1\).

Every element of \(L_1\) is a singular matrix, where as some elements of \(L_2\) are non-zingular. Hence, \(L_1\) and \(L_2\) are not conjugate.

**Problem 6.** (5pts)
Let \(D_{ij} := E_{ij} - E_{ji}\) for any \(1 \leq i, j \leq n\). Then the matrices \(D_{ij} := E_{ij} - E_{ji}\), with \(1 \leq i < j \leq n\), form a basis for \(\mathfrak{so}_n\). Since \(D_{ij} = -D_{ji}\), the Lie algebra structure on \(\mathfrak{so}_n\) is determined by the identities \([D_{ij}, D_{jl}] = D_{il}\) if \(i, j, l\) are distinct, and \([D_{ij}, D_{kl}] = 0\) if \(i, j, l, k\) are distinct. In particular, these identities tell us that \([\mathfrak{so}_n, \mathfrak{so}_n] = \mathfrak{so}_n\), and hence, \(\mathfrak{so}_n\) is not solvable.
Alternatively, we will show that so\(_n\) is not solvable using Cartan’s criterion as follows. Since \(D_{12} = -[D_{13}, D_{23}] \in [so_n, so_n]\), it suffices to show that \(\kappa(D_{12}, D_{12}) = \text{Tr}(\text{ad}(D_{12})^2) \neq 0\). Note that \(\text{ad}(D_{12})\) takes \(D_{1j}\) to \(-D_{2j}\), \(D_{2j}\) to \(D_{1j}\), for any \(j \geq 3\), and all other \(D_{ij}\) with \(i < j\) to 0. Thus

\[
\text{ad}(D_{12})^2(D_{ij}) = \begin{cases} 
-D_{ij}, & \text{if } i = 1 \text{ or } 2, \text{ and } j > 2 \\
0, & \text{if } (i,j) = (1,2) \text{ or } 3 \leq i < j \leq n
\end{cases}
\]

In the basis \(D_{ij}, 1 \leq i < j \leq n\), \(\text{ad}(D_{12})^2\) is represented by a diagonal matrix, with the entries corresponding to \(D_{1j}\) and \(D_{2j}\) equal to \(-1\), for \(3 \leq j \leq n\), and all other entries equal to 0. Thus

\[
\kappa(D_{12}, D_{12}) = \text{Tr}(\text{ad}(D_{12})^2) = -2(n-2) \neq 0,
\]

as desired.