Problem Set 5. Due Tuesday, March 20.


In Problems 1, 2 and 3 below, $F$ will denote a field.

**Problem 1.** Let \[ A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \] be an $m \times n$ matrix with entries in $F$. Consider the $m$ linear polynomials \[ g_1 = a_{11}x_1 + \cdots + a_{1n}x_n, \] \[ \vdots \] \[ g_m = a_{m1}x_1 + \cdots + a_{mn}x_n \] in $F[x_1, \ldots, x_n]$. Let $E$ be an echelon form of $A$ and $R$ be a reduced echelon form of $A$, and $I \subset F[x_1, \ldots, x_n]$ be the ideal generated by $g_1, \ldots, g_m$. Assume that $I \neq (0)$, i.e., $g_i \neq 0$ for some $i$. Show that

(a) the linear polynomials associated to the non-zero rows of $E$ form a minimal Gröbner basis for $I$, and

(b) the linear polynomials associated to the non-zero rows of $R$ form a reduced Gröbner basis for $I$.

(c) Use part (a) to show that the pivot columns in $E$, i.e., the columns containing leading entries, are uniquely determined by $A$.

(d) Show that the reduced echelon form $R$ of $A$ is is unique.

**Problem 2.** Let $I = (g_1, \ldots, g_m) \subset F[x]$. Show that a minimal Gröbner basis for $I$ consists of one element $h$, where $h$ is the monic generator of $I$. If $m = 2$, explain how the Euclidean algorithm (which is used to find $h$ from $g_1$ and $g_2$) is related to Buchberger’s algorithm.

Remark: In this case there is a unique a minimal Gröbner basis, and this minimal Gröbner basis is automatically reduced.

**Problem 3.** Let $G$ be the subgroup of the symmetric group $S_4$ generated by the 2-cycles $(12)$ and $(34)$ and let $R = F[x_1, x_2, x_3, x_4]^G$ be the ring of $G$-invariant polynomials in four variables. Show that $f_1 = x_1 + x_2$, $f_2 = x_1x_2$, $f_3 = x_3 + x_4$ and $f_4 = x_3x_4$ form a Khovanskii basis (SAGBI) for $R$ with respect to the lexicographic monomial order.

**Problem 4.** Chapter 3, Exercise 5.

**Problem 5.** Chapter 3, Exercise 12 (i), (ii), (iii) only.

**Problem 6.** Chapter 5, Exercise 4. Suggestion: Work out the details for the counterexample suggested in the problem. Start by finding $m = \eta \cap A$ in this example.

**Problem 7.** Chapter 5, Exercise 6.

**Problem 8.** Chapter 5, Exercise 7.