Problem Set 4. Due Tuesday, March 6.

**Problem 1.** Chapter 6, Exercise 1.

**Problem 2.** We showed in class that if $R$ is a Noetherian ring, then so is the power series ring $R[[x]]$. This is a variant of the Hilbert Basis Theorem. The purpose of this exercise is to give an alternative proof of this result.

(a) Let $R$ be a ring and $P$ be a prime ideal of $R[[x]]$. Let $Q \subset R$ be the set of elements of the form $f(0)$, as $f(x)$ ranges over $P$. Check that $Q$ is an ideal of $R$. If $Q$ is generated by $n$ elements as an ideal of $R$, show that $P$ can be generated by $m$ elements as an ideal of $R[[x]]$, where $m = n$ if $x \not\in P$ and $m = n + 1$, if $x \in P$.

(b) Combine part (a) with Problem 1 to show that if $R$ is Noetherian, then so is $R[[x]]$.

**Problem 3.** (a) Let $R$ be a Noetherian ring and $N$ be the nilradical of $R$. Show that there exists a positive integer $n$ such that

$$a_1 \cdots a_n = 0$$

for every $a_1, \ldots, a_n \in N$.

**Problem 4.** Chapter 7, Exercise 2.

**Problem 5.** Let $F$ be a field and $F[x, y]$ be a polynomial ring in two variables over $F$, and $M_i = x^i y \in F[x, y]$. Show that the $F$-algebra $R = F[M_1, M_2, \ldots]$ of $F[x, y]$ cannot be generated by a finite number of elements (as an $F$-algebra).

Moral: While ideals of $F[x, y]$ are finitely generated (as ideals), $F$-subalgebras of $F[x, y]$ do not need to be finitely generated (as $F$-algebras).

**Problem 6.** Chapter 7, Exercise 4(i).

**Problem 7.** Chapter 7, Exercise 8.

**Problem 8.** Chapter 7, Exercise 10.