Answer key.

I went over Problems 1, 3 and 6 in class. Here are the answers to the remaining problems.

Problem 2. Consider the ternary (i.e., 3-ary) code $C$ of length 5, consisting of all words $(a_1, a_2, a_3, a_4, a_5)$ such that $a_1 + a_2 + a_3 + a_4 + a_5 \equiv 0 \pmod{3}$.

(a) How many words does $C$ have?
(b) Find the minimal distance of $C$.

Answer: $C$ has 81 words.
Reason: $a_1, \ldots, a_4$ can be arbitrary; there are $3^4 = 81$ ways to choose them. $a_5$ is completely determined by $a_1, \ldots, a_4$.

(b) $d(C) = 2$.
To see that $d(C) \leq 2$, note that $(0, 0, 0, 0, 0)$ and $(1, 2, 0, 0, 0)$ are both in $C$, and are at distance 2 from each other.

To see that $d(C) \neq 1$, we argue by contradiction. Assume that there are two codewords in $C$, $(a_1, a_2, a_3, a_4, a_5)$ and $(b_1, b_2, b_3, b_4, b_5)$, at distance 1. This means that they differ in exactly one position, say, in position $i$. Subtracting

$$a_1 + a_2 + a_3 + a_4 + a_5 \equiv 0 \pmod{3}$$

from

$$b_1 + b_2 + b_3 + b_4 + b_5 \equiv 0 \pmod{3},$$

we see that $b_i - a_i = 0$ in $\mathbb{Z}_3$, contradicting our assumption that $a_i \neq b_i$.

Problem 4. Is there a binary code of length $n = 6$ and size $M = 10$ which corrects 1 error? Explain your answer.

Answer: By the Hamming bound, $A_2(6, 3) \leq \frac{2^6}{1 + 6} = \frac{64}{7} < 10$, so no such code can exist.

Problem 5. Find the missing ISBN digit.

(a) 123?111111
(b) 123123123?

Answer: (a) Solve

$$1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 + 4 \cdot (?) + 5 \cdot 1 + 6 \cdot 1 + 7 \cdot 1 + 8 \cdot 1 + 9 \cdot 1 + 10 \cdot 1 \equiv 0 \pmod{11}$$

This simplifies to

$$1 + 4 + 9 + 4 \cdot (?) + 5 + 6 + 7 + 8 + 9 + 10 \equiv 0 \pmod{11}$$

and thus $4 \cdot (?) + 4 \equiv 0 \pmod{11}$ or equivalently,

$$4 \cdot ((?) + 1) \equiv 0 \pmod{11}.$$
Multiplying both sides by $4^{-1}$, we see that $(?) \equiv -1 \equiv 10 \pmod{11}$. So, the missing digit is $X$.

(b) Solve

$$1 \cdot 1 + 2 \cdot 2 + 3 \cdot 3 + 4 \cdot 1 + 5 \cdot 2 + 6 \cdot 3 + 7 \cdot 1 + 8 \cdot 2 + 9 \cdot 3 + 10 \cdot (?) \equiv 0 \pmod{11}$$

to get $(?) \equiv 8 \pmod{11}$. Thus the missing digit is 8.

**Problem 7.** (a) What does the Hamming bound tell us about $A_2(10,3)$? Same question for (b) the Singleton bound, (c) the Plotkin bound and (d) the Gilbert-Varshamov bound.

**Answer:** (a) $A_2(10,3) \leq \frac{2^{10}}{1+10} = \frac{1024}{11} = 93.1$. Since $A_2(10,3)$ is an integer, this tells us that $A_2(10,3) \leq 93$.

(b) $A_2(10,3) \leq 2^{10-3+2} = 2^8 = 256$.

(c) The Plotkin bound does not apply, because $2d = 2 \cdot 3 < 10 = n$.

(d) $A_2(10,3) \geq \frac{2^{10}}{|B_2(0)|}$, where

$$|B_2(0)| = 1 + \binom{10}{1} + \binom{10}{2} = 1 + 10 + 45 = 56.$$  

Thus $A_2(10,3) \geq \frac{2^{10}}{56} = 18.3$ or, taking into account the fact that $A_2(10,3)$ is an integer,

$$A_2(10,3) \geq 19.$$