(1) How many errors can be detected with the following \(q\)-ary codes? How many errors can be corrected? Explain your answers.

(a) \(C_1 = \{(0, 0, 0, 0, 1), (0, 1, 1, 1), (1, 1, 1, 0, 0)\}\). Here \(q = 2\).

(b) \(C_2 = \{(0, 1, 2, 0, 1, 2), (2, 1, 0, 2, 1, 0), (2, 2, 2, 2, 2, 2)\}\). Here \(q = 3\).

(c) \(C_3 = \{(0, 1, 2, 3, 4, 5, 6), (1, 2, 3, 4, 5, 6, 0), (2, 3, 4, 5, 6, 0, 1), (3, 4, 5, 6, 0, 1, 2), (4, 5, 6, 0, 1, 2, 3), (5, 6, 0, 1, 2, 3, 4), (6, 0, 1, 2, 3, 4, 5)\}\). Here \(q = 7\).

(2) Assume the code \(C_1\) from Problem 1 was used in transmission, and the following words were received. Decode each of these words using the nearest neighbour decoding algorithm. (The incomplete decoding version: if there is more than one nearest neighbour, declare an error.)

(a) \((0, 0, 1, 1, 1)\), (b) \((1, 1, 0, 0, 0)\), (c) \((1, 1, 1, 1, 1)\), (d) \((1, 0, 1, 0, 1)\).

Recall that the triangle inequality for the Hamming distance says that
\[
d(a, b) + d(b, c) \leq d(a, c).
\]
Here \(a\), \(b\) and \(c\) are \(q\)-ary words of length \(n\). We will say that \(b\) lies between \(a\) and \(c\) if equality holds in the above formula, i.e.,
\[
d(a, b) + d(b, c) = d(a, c).
\]
The purpose of the next four exercises is to discover and prove a formula for the number of words that lie between \(a\) and \(c\).

(3) How many words lie between \(a\) and \(a\)?

(4) Assume \(q = 2\) and \(n = 3\). How many words lie

(i) between \((0, 0, 0)\) and \((1, 1, 1)\)?

(ii) between \((0, 0, 0)\) and \((1, 1, 0)\)?

(iii) between \((0, 0, 0)\) and \((1, 0, 0)\)?

(5) Suppose \(q = 2\), \(a\) and \(c\) are binary words of length \(n\) and \(d(a, c) = d\). Based on your answers in Problems 3 and 4, guess a formula for the number of binary words of length \(n\) lying between \(a\) and \(c\). Prove this formula.

(6) We now allow \(q\) to be arbitrary and ask the same question as in Problem 5. Given \(q\)-ary words \(a\) and \(b\) of length \(n\) and at Hamming distance \(d\), how many \(q\)-ary words of length \(n\) lie between \(a\) and \(c\)? Prove your answer.

(7) How many binary words of length 6 are at Hamming distance

(a) at Hamming distance 6 from \((1, 0, 1, 0, 1, 0)\)?

(b) at Hamming distance 5 from \((1, 0, 1, 0, 1, 0)\)?

(8) Is the code \(C_3\) in Problem 1 equivalent to the 7-ary repetition code
\[
C_4 = \{(0, ..., 0), (1, ..., 1), ..., (6, ..., 6)\}
\]
of length 7? Prove your answer.