Problem Set 6. Due in class Thursday, December 1.
Math 322, Fall 2016

1. Suppose a $p$-group $G$ acts on a finite set $X$. If this action has $n$ fixed points in $X$, show that $n \equiv |X| \pmod{p}$.
   (Recall that $x \in X$ is called a fixed point if $g \cdot x = x$ for every $g \in G$.)

2. Let $f : G_1 \to G_2$ be a surjective (i.e., onto) homomorphism of finite groups, $H_2$ be a subgroup of $G_2$ and $H_1 = f^{-1}(H_2) = \{g_1 \in G_1 \mid f(g_1) \in H_2\}$ be the preimage of $H_2$ in $G_1$.
   (a) Show that $|H_1| = |H_2| \cdot |\text{Ker}(f)|$. Here $\text{Ker}(f)$ denotes the kernel of $f$.
   (b) If $H_2$ is a normal subgroup of $G_2$, show that $H_1$ is a normal subgroup of $G_1$.

3. Show that there does not exist a surjective (i.e., onto) homomorphism $f : S_n \to C_p$ for any integer $n \geq 2$ and any prime number $p \geq 3$. Here $C_p$ denote the cyclic group of order $p$.
   Hint: The cases, where $n = 2$, 3 and 4 require special care.

4. Show that $A_4$ is the only subgroup of $S_4$ of order 12.

5. Let $G$ be a group and $H$ be a normal subgroup.
   (a) If $|H| = 2$, show that $H$ is central in $G$. That is, $H \subset Z(G)$.
   (b) Give an example of a group $G$ and a normal subgroup $H \triangleleft G$ such that $H$ is not central in $G$.

6. Let $G$ be a subgroup of $S_n$. Suppose $G$ contains an odd permutation.
   (a) Show that $|G|$ is even, i.e., $|G| = 2k$ for some positive integer $k$.
   (b) Show that exactly $k$ permutations in $G$ are even and the other $k$ are odd.

7. (a) Prove that the alternating group $A_4$ has exactly one Sylow 2-subgroup.
   (b) Prove that $A_5$ has exactly five Sylow 2-subgroups.

8. Let $|G|$ be a finite group of order $n$. For each prime $p$ dividing $n$, let $H_p$ denote the Sylow $p$-subgroup of $G$. Let $S$ be the union of the subgroups $H_p$ over all primes $p$ dividing $n$. Show that $S$ generates $G$, i.e., $\langle S \rangle = G$.
   Hint: What does Lagrange’s theorem say about the order of $\langle S \rangle$?

9. Suppose $G$ has five conjugacy classes with 1, 4, 5, 5 and 5 elements, respectively.
   (a) Does $G$ have a normal subgroup of order 4?
   (b) Does $G$ have a normal subgroup of order 5?

10. Let $G$ be a group, $H \triangleleft G$ be a normal subgroup and $S$ be a Sylow subgroup of $G$. Show that $H \cap S$ is a Sylow subgroup of $H$. 