Problem Set 4. Due in class Thursday, November 3.
Math 322, Fall 2016

(1) Let $G$ be a cyclic group of order $pqr$, where $p$, $q$ and $r$ are distinct primes. How many subgroups does $G$ have (counting the trivial subgroup $\{e\}$ and $G$ itself).

(2) Let $G$ be a finite group, $d \geq 1$ be an integer, and $X = \{a \in G \mid o(a) = d\}$.

(a) Define a relation on $X$ as follows: $(a, b) \in R$ if $<a> = <b>$, i.e., the cyclic subgroup generated by $a$ is the same as the cyclic subgroup generated by $b$. Show that $R$ is an equivalence relation.

(b) Show that the number of elements of order $d$ in $G$ is a multiple of $\phi(d)$. Here $\phi$ denotes the Euler $\phi$-function.

(3) Show that a homomorphism of groups $f : G \to H$ is one-to-one if and only if $\text{Ker}(f) = \{e\}$.

(4) Recall that the centre $Z(G)$ of a group $G$ is defined as follows:

$$Z(G) := \{h \in G \mid hg = gh \text{ for every } g \in G\}.$$ 

In the previous assignment you showed that $Z(G)$ is a subgroup of $G$. Now show that any subgroup $K \subset Z(G)$ is normal in $G$.

(5) Suppose $G$ is a group, $H$ is a subgroup and $g$ is an element of $G$. In the previous assignment you showed that $H^g := \{ghg^{-1} \mid h \in H\}$ is a subgroup of $G$. Now show that the intersection $K := \bigcap_{g \in G} H^g$ is a normal subgroup of $G$.

(6) Show that a subgroup $H$ of $G$ is normal if and only if it has the following property: for any $x, y \in G$ such that $xy \in H$, we also have $yx \in H$.

Hint: This is an "if and only if statement". You need to show that (i) every normal subgroup of $G$ has the above property, and (ii) every subgroup of $G$ which has this property is normal.

(7) Let $G$ be a group and $H$ be a normal subgroup of $G$. Assume that $H$ is finite and cyclic. Show that every subgroup of $H$ is also normal in $G$.

(8) Let $G$ be a finite group and $H$ be a subgroup of index $[G : H] = n$. Recall that $[G : H]$ is the number of left cosets of $H$ in $G$.

(a) If $H$ is a normal subgroup of $G$, show that $x^n \in H$ for every $x \in G$.

(b) Give an example, showing that if $H$ is not assumed to be normal, the assertion of part (a) may fail.

(9) Show that $V = \{\text{id}, (1 \ 3)(2 \ 4), (1 \ 4)(2 \ 3), (1 \ 2)(3 \ 4)\}$ is a normal subgroup of $S_4$.

(10) Suppose $G$ is a finite group, and $H$ is a normal subgroup of $G$. Assume $|H| = m$ and $[G : H] = n$. If $\gcd(m, n) = 1$, show that $|H|$ is the only subgroup of $G$ of order $m$.