Problem Set 3. Due in class Tuesday, October 18.
Math 322, Fall 2016

(1) Give an example of a set $X$ and a relation $R$ on $X$ such that $R$ is symmetric and transitive but not reflexive.

(2) Let $G$ be a group. Show that $Z(G) := \{ h \in G \mid hg = gh \text{ for every } g \in G \}$ is a subgroup of $G$. This subgroup is called the centre of $G$.

(3) Show that $Z(S_n) = \{ \text{id} \}$ for every $n \geq 3$.
   Hint: You need to show that if $\text{id} \neq \sigma \in S_n$, then $\sigma \notin S_n$. Equivalently, for every such $\sigma$ there exists a $\tau \in S_n$ such that $\tau \sigma \neq \sigma \tau$, i.e., $\tau \sigma \tau^{-1} \neq \sigma$. Use the cycle decomposition of $\sigma$ to choose $\tau$. Make sure your solution uses the assumption that $n \geq 3$.

(4) Prove that $S_n$ is generated by the 2-cycles $(1 \ 2), (1 \ 3), \ldots, (1 \ n)$.

(5) Let $\sigma = \left( \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 4 & 5 & 6 & 7 & 2 & 1 \end{array} \right) \in S_7$.
   (a) Decompose $\sigma$ as a product of disjoint cycles.
   (b) Is $\sigma$ even or odd?

(6) Let $\sigma \in S_7$ be as in Problem 5. What is $\sigma^{750}$?
   Hint: Use the cycle decomposition of $\sigma$.

(7) Show that $\text{sgn}(\sigma) = \text{sgn}(\sigma^{-1})$ for every $\sigma \in S_n$. Here $\text{sgn}(\sigma)$ denotes the sign of $\sigma$.

(8) Let $\sigma \in S_n$ be given by $\sigma(i) = n + 1 - i$, for $i = 1, \ldots, n$. What is the sign of $\sigma$?
   Hint: The answer depends on $n$.

(9) If $f : G \to H$ is a homomorphism of groups, and $a \in G$ is an element of finite order, then the order of $f(a)$ divides the order of $a$.

(10) Suppose $\sigma$ a a group $G$ has two subgroups, $H$ of order 11 and $K$ of order 17. Show that $H \cap K = \{ e \}$.