Exercise set 1

1. Induction on \(n\).

Base case, \(n = 1\). We check that \(1 \times 2 = \frac{1(2)(3)}{3}\).

Induction step, \(n \Rightarrow n + 1\): Suppose that the result is true for some \(n \geq 1\), let us prove that it is true for \(n + 1\).

\[
\sum_{k=1}^{n+1} k(k+1) = \sum_{k=1}^{n} k(k+1) + (n+1)(n+2)
\]

\[
= \frac{n(n+1)(n+2)}{3} + (n+1)(n+2)
\]

by induction hypothesis

\[
= \frac{n(n+1)(n+2) + 3(n+1)(n+2)}{3}
\]

\[
= \frac{(n+1)(n+2)(n+3)}{3}
\]

as desired. Therefore the result is true for \(n + 1\). By induction on \(n\) we proved that it holds for all \(n \in \mathbb{N}\).

2. We proceed by induction on \(n\). Base case, \(n = 0\). We have \((1 + x)^0 = 1 \geq 1 + 0x\).

Induction step \(n \Rightarrow n + 1\). Suppose that the result is true for some \(n \geq 1\), let us prove that it is true for \(n + 1\).

\[
(1 + x)^{n+1} = (1 + x)^n(1 + x)
\]

\[
\geq (1 + nx)(1 + x)
\]

by induction hypothesis

\[
= 1 + (n + 1)x + nx^2
\]

\[
\geq 1 + (n + 1)x
\]

so the inequality is true for \(n + 1\). The result is therefore proved by induction on \(n\).

3. We proceed by strong induction on \(n\).

Base cases \(n = 0\), \(n = 1\) and \(n = 2\) are immediate.

Induction step \(n-3,n-2,n-1 \Rightarrow n\): Suppose the result is true for all \(k \leq n-1\) with \(n \geq 3\). Let us prove that it is true for \(n\).

\[
a_n = a_{n-1} + a_{n-1} + a_{n-3}
\]

since \(n \geq 3\)

\[
\leq 3^{n-1} + 3^{n-2} + 3^{n-3}
\]

\[
\leq 3^{n-1} + 3^{n-1} + 3^{n-1}
\]

by induction hypothesis

\[
\leq 3(3^{n-1}) = 3^n.
\]
Therefore, we proved by induction that the inequality hold for every $n \in \mathbb{N}$.

4. Let us prove the hint first. Let $x, y$ be two positive real numbers. Since squares are always positive, we have that

$$0 \leq \left( \sqrt{x} - \sqrt{y} \right)^2 = x - 2\sqrt{xy} + y,$$

therefore $2\sqrt{xy} \leq x + y$.

Now set $A_1 = a_1 + \cdots + a_n + a_{n+1}$ and $B_n = \frac{1}{a_1} + \cdots + \frac{1}{a_n}$. We want to prove that $A_n B_n \geq n^2$ by induction on $n$.

Base case, $n = 1$. We have $A_1 B_1 = \left( \frac{1}{a_1} \right) = 1 \geq 1$.

Induction step, $n \Rightarrow n + 1$. Assume $A_n B_n \geq n^2$ for some $n \geq 1$. We want to show that $A_{n+1} B_{n+1} \geq (n + 1)^2$.

\[
A_{n+1} B_{n+1} = (A_n + a_{n+1}) \left( B_n + \frac{1}{a_{n+1}} \right) + 1
\]

\[
= A_n B_n + a_{n+1} B_n + \frac{1}{a_{n+1}} A_n + 1
\]

\[
\geq n^2 + a_{n+1} B_n + \frac{1}{a_{n+1}} A_n + 1 \quad \text{by the induction assumption}
\]

\[
\geq n^2 + 1 + 2\sqrt{a_{n+1} B_n \frac{1}{a_{n+1}} A_n} \quad \text{by the inequality in the hint}
\]

\[
= n^2 + 1 + 2\sqrt{B_n A_n}
\]

\[
\geq n^2 + 1 + 2\sqrt{n^2} \quad \text{by the induction assumption}
\]

\[
= (n + 1)^2,
\]

as desired.

We have thus proved the result by induction for all $n \in \mathbb{N}$.

5. We proceed by induction on $n$.

$n = 1$. We have a $2 \times 2$ square chessboard. Once we remove a piece, there is only room for one L-shaped piece left.

\[
\begin{array}{cccc}
\text{\square} & \text{\square} & \text{\square} & \text{\square} \\
\text{\square} & \text{\square} & \text{\square} & \text{\square} \\
\text{\square} & \text{\square} & \text{\square} & \text{\square} \\
\text{\square} & \text{\square} & \text{\square} & \text{\square}
\end{array}
\]

$n \Rightarrow n + 1$. Assume the result holds true for some $n \geq 1$. Let us prove it is true for $n + 1$. We have a square of size $2^{n+1} \times 2^{n+1}$. Let us cut it in 4 squares all of size $2^n \times 2^n$. The problem is invariant by rotation of the square, so we can assume without loss of generality that the hole is in the top right square. By induction hypothesis, we can fill the top right square with L-shaped pieces.

We can also use the induction hypothesis for the other 3 squares, if we remove a tile in each square, we can fill the rest with the pieces. We choose to remove
the tiles closest to the middle of the chessboard, so we are in the following scenario.

By induction hypothesis, we fill all the 4 squares except the 4 tiles we removed, one in each square. But since the three pieces in the middle form an L-shape, we can fill them with a piece and we’re left with the only hole we started with. We filled the $2^{n+1} \times 2^{n+1}$ chessboard except the missing square with L-shaped pieces that do not overlap.

Therefore, we proved by induction on $n$ that the result is true for all $n \in \mathbb{N}$.

6. Let $a, b$ be integers with $b > 0$. We write the division of $a$ by $b$ as $a = qb + r$ with $q$ an integer and $0 \leq r < r$.

If $0 \leq r \leq \frac{b}{2}$ then we have nothing to do.

If $\frac{b}{2} < r < b$ write $a = (q + 1)b + (r - b)$ and $-\frac{b}{2} < r - b < 0$ which is what we want.

7. First, assume that $a$ and $b$ are coprime, in other words $\gcd(a, b) = 1$. We have that $c$ divides both $ca$ and $cb$ therefore $c$ divides $\gcd(ca, cb)$.

For the converse, since $a$ and $b$ are coprime, by Bezout’s theorem we know that there are $u, v \in \mathbb{Z}$ such that $ua + vb = 1$, therefore $u(ca) + v(cb) = c$. Note that $\gcd(ca, cb)$ divides $ca$ and $cb$ so it divides $u(ca) + v(cb) = c$.

We proved that $c$ divides $\gcd(ca, cb)$ and conversely $\gcd(ca, cb)$ divides $c$. Since both are positive integers we get that $\gcd(ca, cb) = c$ as desired.
Now if \( \gcd(a, b) \neq 1 \), write \( g = \gcd(a, b) \) and \( a = ab, b = \beta g \) with \( \gcd(\alpha, \beta) = 1 \).

We then have

\[
\gcd(ca, cb) = \gcd((cg)\alpha, (cg)\beta) = cg = c\gcd(a, b),
\]

using the previous result with \( \alpha \) and \( \beta \).

8. Take \( t_1, \ldots, t_n \in \mathbb{Z} \) such that \( t_1a_1 + \cdots + t_na_n \) is the smallest positive linear combination of \( a_1, \ldots, a_n \). Write \( d = t_1a_1 + \cdots + t_na_n \).

Clearly since \( \gcd(a_1, \ldots, a_n) \) divides \( a_1 \) to \( a_n \), it also divides the linear combination, so \( \gcd(a_1, \ldots, a_n) \) divides \( d \).

For the converse, let us prove that \( d \) divides \( a_1 \) to \( a_n \). Let \( i \in \{1, \ldots, n\} \). Write the Euclidean division of \( a_i \) by \( d \) as \( a_i = qd + r \) with \( 0 \leq r < d \).

We have

\[
q = a_i - qd = a_i - q(t_1a_1 + \cdots + t_na_n) = -(qt_1)a_1 - \cdots - (qt_i-1)a_{i-1} + (1 - qt_i)a_i - (qt_{i+1})a_{i+1} - \cdots - (qt_n)a_n.
\]

This gives us a linear combination of the \( a_i \)’s equals to \( r \), and since \( 0 \leq r < d \), by minimality of the linear combination \( t_1a_1 + \cdots + t_na_n \) we must have that \( r = 0 \).

Therefore, \( d \) divides \( a_i \) for all \( i \in \{1, \ldots, n\} \) so \( d \) divides \( \gcd(a_1, \ldots, a_n) \). Since both \( d \) and \( \gcd(a_1, \ldots, a_n) \) are nonnegative, and divide each other, they must be equal, as desired.

9. We have to find three numbers such that no prime number is a divisor of all of them but if we take three of them, they will have a common prime divisor.

To that extent, we can take any four prime numbers \( p_1, p_2, p_3, p_4 \) and define \( a = p_1p_2p_3, b = p_2p_3p_4, c = p_3p_4p_1 \) and \( d = p_4p_1p_2 \).

Let us do it with the primes \( 2, 3, 5, 7 \). Take \( a = 2 \times 3 \times 5 = 30, b = 3 \times 5 \times 7 = 105, c = 5 \times 7 \times 2 = 70 \) and \( d = 7 \times 2 \times 3 = 42 \).

We can check \( \gcd(30, 105, 70, 42) = 1 \), but \( \gcd(30, 105, 70) = 5, \gcd(105, 70, 42) = 7, \gcd(70, 42, 30) = 2, \gcd(42, 30, 105) = 3 \).

10. Let us use the Fundamental Theorem of Arithmetic. Write \( a = \prod_{p \text{ prime}} p^{\alpha_p} \) and \( b = \prod_{p \text{ prime}} p^{\beta_p} \) with only finitely many \( \alpha_p \) and \( \beta_p \) nonzero. We want to prove that \( a \) divides \( b \), in other words that \( \alpha_p \leq \beta_p \) for all prime \( p \).

We can treat the products as finite products since only finitely many factors are different from 1. Therefore,

\[
a^3 = \left( \prod_{p \text{ prime}} p^{\alpha_p} \right)^3 = \prod_{p \text{ prime}} p^{3\alpha_p},
\]

\[
b^2 = \left( \prod_{p \text{ prime}} p^{\beta_p} \right)^2 = \prod_{p \text{ prime}} p^{2\beta_p}.
\]
We know that $a^3$ divides $b^2$, therefore for all prime $p$, we have $3\alpha_p \leq 2\beta_p$ and so $\alpha_p \leq \frac{3}{2}\alpha_p \leq \beta_p$ which implies that $a$ divides $b$. 