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This is one of the axioms of the natural numbers.
Suppose $a$ is a fixed integer, and $P(a)$ is an assertion, that is either true or false for each integer $n \geq a$. We want to prove that $P(n)$ is true for each $n \geq a$.

The Principle of Mathematical Induction tells us that in order to prove this it is enough to

(i) Prove that $P(a)$ is true. This is called “the base case”.

(ii) Prove that for every $n \geq a$, if $P(n)$ is true, then $P(n+1)$ is also true. This is called “the induction step”.

If we can establish (i) and (ii), then property $P(n)$ will be true for every integer $n \geq a$. To see that this proof method is valid, denote the set of non-negative integers $m$ such that $P(a+m)$ is false by $S$. The Well Ordering Principle now tells us that $S$ has to be empty.
Mathematical induction

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If we can establish (i) and (ii), then property $P(n)$ will be true for every integer $n \geq a$. To see that this proof method is valid, denote the set of non-negative integers $m$ such that $P(a + m)$ is false by $S$. The Well Ordering Principle now tells us that $S$ has to be empty. induction to show that $S = \mathbb{N}$. 
1. Show that \(1 + 3 + 5 + \cdots + (2n - 1) = n^2\) for every \(n \geq 1\).

2. Show that \(1 + q + q^2 + \cdots + q^n = \frac{q^{n+1} - 1}{q - 1}\) for any real number \(q \geq 0\).

3. Show that \(2^n > n^2\) for any \(n \geq 5\).

4. Show that \(n\) lines in general position subdivide the plane into \(\frac{n(n+1)}{2} + 1\) regions. Here is \(n\) is an integer \(\geq 1\).

5. Show that \(n^3 - n\) is divisible by 3 for any \(n \geq 0\).
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(1) $P(a)$ is true. This is called “the base case”.

(2) Show that if $P(1), \ldots, P(n)$ are all true for some $n \geq a$, then $P(n + 1)$ is also true. This is called “the induction step”.

1. Show that every integer $n \geq 2$ is either a prime or a product of two or more primes.

2. Any integer amount of postage of 12 cents or more, can be paid using only 3-cent and 5-cent stamps.

3. The $n$th Fibonacci number $a_n$ is defined by the recursive formula $a_1 = a_2 = 1$, $a_{n+2} = a_{n+1} + a_n$. Show that

$$a_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n)$$

for any $n \geq 1$. Here

$$\alpha = \frac{1 + \sqrt{5}}{2} \text{ and } \beta = \frac{1 - \sqrt{5}}{2}.$$
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   \[ a_1 = a_2 = 1, \ a_{n+2} = a_{n+1} + a_n. \] 
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   \[ \alpha = \frac{1 + \sqrt{5}}{2} \text{ and } \beta = \frac{1 - \sqrt{5}}{2}. \]
   Note that \( \alpha \) and \( \beta \) are the roots of the quadratic equation \( x^2 - x - 1 = 0 \).
Strong mathematical induction exercises

1. Show that every integer \( n \geq 2 \) is either a prime or a product of two or more primes.

2. Any integer amount of postage of 12 cents or more, can be paid using only 3-cent and 5-cent stamps.

3. The \( n \)th Fibonacci number \( a_n \) is defined by the recursive formula
   \[ a_1 = a_2 = 1, \ a_{n+2} = a_{n+1} + a_n. \]
   Show that
   \[ a_n = \frac{1}{\sqrt{5}}(\alpha^n - \beta^n) \]
   for any \( n \geq 1 \). Here
   \[ \alpha = \frac{1 + \sqrt{5}}{2} \text{ and } \beta = \frac{1 - \sqrt{5}}{2}. \]

   Note that \( \alpha \) and \( \beta \) are the roots of the quadratic equation \( x^2 - x - 1 = 0 \).
   That is, \( \alpha^2 = \alpha + 1 \) and \( \beta^2 = \beta + 1 \). These formulas will facilitate the induction step.
Remark

The well-ordering principle, the principle of mathematical induction and the principle of strong mathematical induction are all equivalent to each other.