Multiplicative Functions (Sect. 7.2)

An arithmetic function, is a function defined on the set of positive integers. An arithmetic function \( f(n) \) is called multiplicative if \( f(km) = f(k)f(m) \) whenever \( \gcd(k, m) = 1 \).

Example: The Euler \( \phi \)-function is multiplicative.

We will now consider two other examples:

\( \sigma(n) = \text{sum of divisors of } n \).
\( \tau(n) = \text{no. of divisors of } n. \)

Theorem: If \( f(n) \) is a multiplicative function, then \( F(n) = \sum_{d|n} f(d) \) is also multiplicative.
Taking \( f(n) = 1 \) (for every \( n \)), we see that \( \tau(n) = \sum_{d|n} 1 \) is multiplicative.

Taking \( f(n) = n \), we see that
\[
\sigma(n) = \sum_{d|n} d \quad \text{is multiplicative.}
\]

**Theorem:** If \( n = p_1^{d_1} \ldots p_r^{d_r} \) is the prime decomposition of \( n \), \( f(n) \) is a multiplicative function, then
\[
f(n) = f(p_1^{d_1}) \ldots f(p_r^{d_r})
\]

**Proof:** Induction on \( r \), as in the case of \( \phi(n) \).

**Corollary:**
\[
\sigma(n) = \frac{p_1^{d_1+1} - 1}{p_1 - 1} \ldots \frac{p_r^{d_r+1} - 1}{p_r - 1}
\]
\[
\tau(n) = (d_1+1) \ldots (d_r+1).
\]
Example: \( n = 100 = 2^2 \cdot 5^2 \). Then

\[ 6(n) = \frac{2^3 - 1}{2 - 1} \cdot \frac{5^3 - 1}{5 - 1} = 7 \cdot 31 = 217 \]

\[ \tau(n) = 3 \cdot 3 = 9 \]