Exam rules:
- No calculators, open books or notes are allowed.
- Show all your work.
- Use opposite empty pages if needed.
- There are 5 problems in this exam. The first 4 problems are worth 10 marks each, the last problem is worth 3 marks.

Formulas:
\[
\sin \frac{\pi}{6} = \cos \frac{\pi}{3} = \frac{1}{2}, \\
\sin \frac{\pi}{3} = \cos \frac{\pi}{6} = \frac{\sqrt{3}}{2}, \\
\sin \frac{\pi}{4} = \cos \frac{\pi}{4} = \frac{\sqrt{2}}{2}.
\]

Cauchy-Riemann equations:
\[u_x = u_y, \quad u_y = -u_x.\]
PROBLEM 1. (a) Write the following number in the form \( a + bi \):

\[
(i - 1)^{10}.
\]

\[
\left( \sqrt{2} \ e^{\frac{3\pi i}{4}} \right)^{10} = \left( \sqrt{2} \right)^{10} \ e^{\frac{30\pi i}{4}} = 32 \ e^{\frac{15\pi i}{2}} = 32 \ e^{\frac{3\pi i}{2}}.
\]

(b) Write the following number in the form \( a + bi \):

\[
\exp(1 + \frac{\pi i}{3}) + \exp(1 - \frac{\pi i}{3}).
\]

\[
e^{i} \left( \cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right) + e^{i} \left( \cos \left(-\frac{\pi}{3} \right) + i \sin \left(\frac{\pi}{3} \right) \right)
\]

\[
= 2 \cdot e \cdot \cos \frac{\pi}{3} = 2 \cdot e \cdot \frac{1}{2} = e.
\]
PROBLEM 2. (a) Sketch the set of all complex numbers \( z \) such that 
\[
\frac{\pi}{3} \leq \text{Arg}(z) \leq \frac{2\pi}{3} \quad \text{and} \quad |z| \geq 2.
\]

(b) Sketch the image of the set in part (a) by the function \( f(z) = -iz^2 \).

First apply \( z \mapsto z^2 \), then multiply with \(-i\):
PROBLEM 3. (a) Find all third roots of $z = -8i$. Leave your final answers in the form $a + bi$.

$$z = 8 \cdot e^{-i \frac{\pi}{2}} \cdot e^{-i \frac{2\pi k}{3}} = 2 \cdot e^{-i \frac{2\pi k}{3}}$$

$k = 0$: 

$$2 \cdot e^{-i \frac{2\pi}{3}} = 2 \left( \cos \frac{2\pi}{3} - i \sin \frac{2\pi}{3} \right) = -\sqrt{3} - i$$

$k = 1$: 

$$2 \cdot e^{-i \frac{4\pi}{3}} = 2 \cdot e^{-i \frac{2\pi}{3}} = -\sqrt{3} - i$$

$k = 2$: 

$$2 \cdot e^{-i \frac{6\pi}{3}} = 2 \cdot e^{-i \frac{2\pi}{3}} = -\sqrt{3} - i$$

(b) Find all solutions to the equation

$$z^3 + (i - 1)z^2 - iz = 0.$$ 

Your final answers should be in the form $a + bi$.

$$Z = 0$$

$$Z^2 + (i - 1)Z - i = 0$$

$$Z = \frac{i - 1}{2} \pm \sqrt{\left(\frac{i - 1}{2}\right)^2 + i}$$

$$Z = \frac{i - 1}{2} \pm \sqrt{-\frac{1}{2} i + i}$$

$$Z = \frac{i - 1}{2} \pm \sqrt{\frac{1}{2} i}$$

$$Z = \frac{i - 1}{2} \pm \frac{1}{\sqrt{2}} \sqrt{i}$$

$$Z = \frac{i - 1}{2} \pm \frac{1}{\sqrt{2}} \left( \frac{\sqrt{2}}{2} + i \frac{\sqrt{2}}{2} \right) = \frac{i - 1}{2} \pm \frac{1}{2} + \frac{1}{2} i$$

$$Z_1 = 1 + i$$

$$Z_2 = -i$$
PROBLEM 4. Let
\[ v(x, y) = x^3(y + 1) - xy^2(y + 3). \]
(a) Find a function \( u(x, y) \), such that \( f(x + iy) = u(x, y) + iv(x, y) \) is analytic.
\[
\begin{align*}
\psi_x &= \psi_y = x^3 - 3x y^2 - 6x y \\
\psi &= \frac{x^4}{4} - \frac{3x^2 y^2}{2} - 3x y^2 - \frac{x^2 y}{2} + g(y) \\
\phi_x &= -3x y^2 - \frac{3y^4}{4} + g'(y) \\
\phi &= -\phi_x = -\left(3x^2 y + 3x^2 y^3 - 3y^3 - 3y^4\right) \\
g'(y) &= y^3 \text{ and } g(y) = \frac{y^4}{4} + y^3 + C \\
u(x, y) &= \frac{x^4}{4} - \frac{3x^2 y^2}{2} - 3x y^2 + \frac{x^2 y}{2} + y^3 + C.
\end{align*}
\]
(b) Find the second derivative of \( f(x + iy) = u(x, y) + iv(x, y) \) and express it in terms of \( z = x + iy \). (The second derivative \( f''(z) \) should be a degree two polynomial in \( z \).)
\[
\begin{align*}
\psi'_{(x + iy)} &= \psi_x + i\psi_y \\
\psi''_{(x + iy)} &= \psi_{xx} + i\psi_{xy} \\
&= 3x^2 - 3y^2 - 6y + i\left(6xy + 6x\right) \\
&= \left[3x^2 - 3y^2 + i6xy\right] + \left[-6y + i6x\right] \\
&= \sqrt{3z^2 + 6i\bar{z}}.
\end{align*}
\]
PROBLEM 5. An analytic function $f(z)$ is defined on a domain $D$ and has its range a subset of the line $y = x$ on the complex plane. Find all such functions $f(z)$.

\[ f(x + iy) = u(x, y) + iv(x, y), \quad u = v \text{ for all } x, y. \]

Cauchy-Riemann:

\[
\begin{align*}
\frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} = u_y = v_y, \\
\frac{\partial u}{\partial y} &= -\frac{\partial v}{\partial x} = -u_x = -v_x.
\end{align*}
\]

\[ \Rightarrow \quad u_x = -u_x, \quad u_x = 0, \quad u_y = 0, \quad v_x = 0, \quad v_y = 0 \]

\[ \Rightarrow \quad u, v \text{ are constants, } \]

\[ f(z) = \text{const}. \]