Harmonic Functions

$u(x,y)$ - real-valued function of two real variables, continuous 2nd order partial derivatives. Harmonic if

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{(Laplace equation)}.$$

Last time: If $f(z)$ is analytic in an open region $D$ of $\mathbb{C}$, and $f(x+yi) = u(x,y) + v(x,y)i$, then $u(x,y)$ and $v(x,y)$ are harmonic.

Conversely, if $u(x,y)$ is harmonic, then there exists a $v(x,y)$ such that $f(z) = u + vi$ is analytic.

$v(x,y)$ is called the harmonic conjugate of $u(x,y)$. Here the domain of definition of $u(x,y)$ is assumed to be a circle or all of $\mathbb{C}$. 
\( v(x,y) \) is unique up to an additive constant (in a domain).

Proof of uniqueness: If \( v_1(x,y) \) and \( v_2(x,y) \) are both harmonic conjugates of \( u(x,y) \), then \( f(z) = u + i v_1 \) are both \( g(z) = u + i v_2 \) analytic.

Then \((f-g)(z) = i(v_1-v_2)\) is also analytic, assumes only pure imaginary values. Thus \(f-g\) is a constant function \( \Rightarrow v_1-v_2 \) is constant, as claimed.

Existence: Example

\[ u(x,y) = 3x^2y - y^3 + x^2 - y^2 \]

\[ \frac{\partial u}{\partial x} = 6xy + 2x \]

\[ \frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 2y \]

\[ \frac{\partial^2 u}{\partial x^2} = 6y + 2 \]

\[ \frac{\partial^2 u}{\partial y^2} = -6y - 2 \]
Let us find the harmonic conjugate of \( u(x,y) \).

Looking for \( v(x,y) \) such that

\[
\frac{\partial v}{\partial x} = - \frac{\partial u}{\partial y} = -3x^2 + 3y^2 + 2y
\]

\[
\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = 6xy + 2x
\]

Anti-differentiate \( v \) with respect to \( y \):

\[
v(x,y) = 3xy^2 + 2xy + \psi(x)
\]

where \( \psi(x) \) is some function of \( x \).

Now match \( \frac{\partial v}{\partial x} \) to \(-3x^2 + 3y^2 + 2y\).

\[
\frac{\partial v}{\partial x} = 3y^2 + 2y + \psi'(x).
\]

Thus \( \psi'(x) = -3x^2 \Rightarrow \psi(x) = -x^3 + C. \)

Answer: \( v(x,y) = 3xy^2 - x^3 + 2xy + C \)

where \( C \) is a real constant.
\[ f = u + iv = (3x^3y - y^3 + x^2 - y^2) + \\
(3xy^2 - x^3 + 2xy)i \]

is analytic. Can we express it as a polynomial in \( z = x + iy \)?

Yes, \( f(z) = -2iz^3 + z^2 \)

---

Level curves of harmonic cons. functions \( u(x, y) \) and \( v(x, y) \) are perpendicular at every point where they meet.

\[ \nabla u = \left( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) \text{ gradient, perpendicular to level curve for } u. \]

\[ \nabla v = \left( \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \right) \text{ gradient for } v. \]

\[ = \begin{pmatrix}
-\frac{\partial u}{\partial y} \\
\frac{\partial u}{\partial x}
\end{pmatrix} \]

\[ \nabla u \cdot \nabla v = 0. \text{ This tells me that level curves for } u, v \text{ are orthogonal.} \]
Example 1 \[ f(z) = z = x + yi. \]
\[ u(x,y) = x \]
\[ v(x,y) = y \]

Example 2 \[ f(z) = z^2 = (x + iy)^2 = (x^2 - y^2) + 2(xy)i. \]
\[ u(x,y) = x^2 - y^2 \]
\[ v(x,y) = 2xy \]
Steady state Temperature

\[ T(x, y) \] 
steady state temperature at \((x, y)\).

Net outflow = 0 
(no heat sources in interior).

\[ - \frac{\partial T}{\partial x} \bigg|_{AB} + \frac{\partial T}{\partial x} \bigg|_{CD} + \frac{\partial T}{\partial y} \bigg|_{BC} - \frac{\partial T}{\partial y} \bigg|_{AD} \]

\[ \frac{\partial^2 T}{\partial x^2} \cdot \Delta S + \frac{\partial^2 T}{\partial y^2} \cdot \Delta S = 0 \]

\[ \frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0. \]

\[ T(x, y) \] 
is harmonic
Observations: $T(x, y)$ in interior is completely determined by the values on the boundary.

Maximum principle: Maximal value of $T(x, y)$ cannot occur in the interior.

Polynomial Functions

$p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$

$a_0, \ldots, a_n$ complex numbers.

If $a_n \neq 0$, we say that $p(z)$ is a polynomial of degree $n$.

Taylor form at $z = z_0$

$p(z) = b_0 + b_1 (z - z_0) + \cdots + b_n (z - z_0)^n$
Taylor form always exists. Set \( W = z - z_0 \).

Expand

\[
p(z) = p(W + z_0) = b_0 + b_1 W + \ldots + b_n W^n = b_0 + b_1 (z - z_0) + \ldots + b_n (z - z_0)^n.
\]

Another way to construct Taylor form of \( p(z) \): compare derivatives at \( z = z_0 \).

\[ p(z_0) = b_0 \]

\[ p'(z) = b_1 + 2b_2 (z - z_0) + 3b_3 (z - z_0)^2 + \ldots \]

\[ p'(z_0) = b_1 \]

\[ p''(z) = 2b_2 + 6b_3 (z - z_0) + \ldots \]

\[ p''(z_0) = 2b_2 \]

Continue recursively.

Here \( p^{(k)} \) denotes the \( k \)th derivative of \( p \).
Proposition: Let \( p(z) \) be a polynomial of degree \( n \geq 1 \). Then

1. \( z_0 \) is a root of \( p(z) \) if and only if \( p(z) = (z-z_0)q(z) \) for some polynomial \( q(z) \) of degree \( n-1 \).

2. \( p(z) \) has at most \( n \) complex roots.

Proof of (1): \( z_0 \) is a root if and only if \( b_0 = 0 \), i.e.

\[
p(z) = b_1(z-z_0) + b_2(z-z_0)^2 + \cdots + b_n(z-z_0)^n
\]

Equivalently, \( p(z) = (z-z_0)q(z) \), where \( q(z) \) is a polynomial of degree \( n-1 \).

2. Apply (1) recursively to write

\[
p(z) = (z-z_1)(z-z_2) \cdots (z-z_k)s(z)
\]
where $z_1, z_2, \ldots, z_k$ are distinct roots of $p(z)$, and $s(z)$ is a polynomial of degree $n-k$. Clearly this cannot be done if $k > n$.

**Example:** $p(z) = z^3 - 1 = (z-1)(z^2 + z + 1)$

$z_1 = 1$, $z_2 = e^{2\pi i/3}$, $z_3 = e^{-2\pi i/3}$

Roots of $p(z) =$ cube roots of $1$

$z_2 = -\frac{1}{2} + \frac{\sqrt{3}}{2} i$

$z_3 = -\frac{1}{2} - \frac{\sqrt{3}}{2} i$

$z^3 - 1 = (z-1)(z-z_2)(z-z_3)$. 