
w = f(z) complex-valued function. ①

Last time: Defined \( \lim_{z \to z_0} f(z) \).

**Theorem 1** Suppose \( \lim_{z \to z_0} f_1(z) = w_1 \)
\[ \lim_{z \to z_0} f_2(z) = w_2 \]

Then

1. \( \lim_{z \to z_0} (f_1 + f_2)(z) = w_1 + w_2 \)
2. \( \lim_{z \to z_0} (f_1 - f_2)(z) = w_1 - w_2 \)
3. \( \lim_{z \to z_0} (f_1 f_2)(z) = w_1 \cdot w_2 \)
4. \( \lim_{z \to z_0} \frac{f_1}{f_2}(z) = \frac{w_1}{w_2} \) if \( w_2 \neq 0 \).
\[ f(x+iy) = u(x,y) + v(x,y)i \quad \text{if } \quad z_0 = x_0 + y_0 i \]

\[
\lim_{z \to z_0} f(z) = a + bi \quad \text{if and only if} \\
\lim_{(x,y) \to (x_0, y_0)} u(x,y) = a \quad \text{and} \quad \lim_{(x,y) \to (x_0, y_0)} v(x,y) = b
\]

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**f(z) is continuous at** \( z_0 \) **if**

\[
\lim_{z \to z_0} f(z) = f(z_0).
\]

**f(z) is continuous at** \( z_0 \) **if and only if**

\( u(x,y) \) **and** \( v(x,y) \) **are both continuous at** \( (x_0, y_0) \).

**Theorem 2:** If \( f_1(z) \) **and** \( f_2(z) \) **are both continuous at** \( z_0 \), **then so are**

\( f_1 + f_2)(z), (f_1 - f_2)(z), (f_1 f_2)(z) \). **Moreover,**

**if** \( f_2(z_0) \neq 0 \), **then** \( \frac{f_1}{f_2}(z) \) **is also continuous at** \( z_0 \).
The complex derivative

\[ f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \]

\[ = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} \]

Also write \( \frac{df}{dz} \mid z = z_0 \) for \( f'(z_0) \)

Example: \( f(z) = \text{Re}(z) \). What is \( f'(z_0) \)?

\[ f'(z_0) = \lim_{\Delta z \to 0} \frac{\text{Re}(z_0 + \Delta z) - \text{Re}(z_0)}{\Delta z} \]

\[ = \lim_{\Delta z \to 0} \frac{\Delta x}{\Delta z} \text{, where } \Delta z = \Delta x + i \Delta y \]

\[ \lim_{\Delta z \to 0} \Delta x = \lim_{\Delta x \to 0} \Delta x = 1 \]

\[ \lim_{\Delta z \to 0} \frac{\Delta x}{\Delta z} = \lim_{\Delta z \to 0} \frac{0}{\Delta z} = 0 \]
Conclusion: In this example, \( \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} \) does not exist. In other words, \( f'(z_0) \) does not exist for any complex number \( z_0 \). The function \( f(z) = \text{Re}(z) \) is not differentiable at any point.

Note \( f(z) = u(x, y) + v(x, y)i \). \( z = x + yi \).

Here \( u(x, y) = x \). Both \( u \) and \( v \) \( v(x, y) = 0 \). have partial derivatives, yet the complex derivative of \( f(z) = \text{Re}(z) \) does not exist.

Another example: \( f(z) = z^2 \).
\[ f'(z_0) = \lim_{\Delta z \to 0} \frac{(z_0 + \Delta z)^2 - z_0^2}{\Delta z} \]
\[ = \lim_{\Delta z \to 0} \frac{z_0^2 + 2z_0 \Delta z + (\Delta z)^2 - z_0^2}{\Delta z} \]
\[ = \lim_{\Delta z \to 0} (2z_0 + \Delta z) = 2z_0. \]

**Theorem 3** If \( f(z) \) and \( g(z) \) are both differentiable at \( z_0 \), then so are \( (f+g)(z) \), \( (f-g)(z) \), \( (f \cdot g)(z) \). If \( g'(z_0) \neq 0 \), the \( \frac{f}{g}(z) \) is also differentiable at \( z_0 \). Moreover,

1. \( (f+g)'(z_0) = f'(z_0) + g'(z_0) \)
2. \( (f \cdot g)'(z_0) = f'(z_0) g(z_0) + f(z_0) g'(z_0) \)
3. \( \left( \frac{f}{g} \right)'(z_0) = \frac{f'(z_0) g(z_0) - f(z_0) g'(z_0)}{g(z_0)^2} \)
Corollary 1: If \( f(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \) is a polynomial function, then \( f'(z) \) exists for every complex number \( z_0 \). Here \( a_0, \ldots, a_n \) are complex numbers.

Proof: The functions \( f(z) = z \) and \( g(z) = a = \text{constant} \) are both differentiable (check!)

Any polynomial can be obtained from these via addition and multiplication. Now use Theorem 3.

Corollary 2: \( f(z) = \overline{z} \) is not differentiable at any \( z_0 \).

Proof: \( \text{Re}(z) = \frac{1}{2} (z + \overline{z}) \). If \( \overline{z} \) is differentiable, then so is \( \text{Re}(z) \). But we know that \( \text{Re}(z) \) is not.

Exercise: Show that \( \text{Im}(z) \) is not differentiable at any \( z_0 \).
Hint: \( \overline{z} = z - 2 \text{Im}(z)i. \)

Write \( f(x+iy) = u(x,y) + v(x,y)i \)

Theorem 3 Suppose \( f(z) \) is differentiable at \( z_0 \). Then

(a) \[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}
\]
Cauchy–Riemann equations

(b) \( f'(z_0) = \frac{\partial u}{\partial x}(x_0,y_0) + \frac{\partial v}{\partial x}(x_0,y_0)i \)

Example 1: \( f(z) = z \). Here

\( u(x,y) = x \) \quad \text{Cauchy–Riemann} \quad \frac{\partial u}{\partial x} = 1 \quad \frac{\partial v}{\partial x} = 0 \)

\( v(x,y) = y \) \quad \frac{\partial u}{\partial y} = 0 \quad \frac{\partial v}{\partial y} = 1 \)
Example 2 \( f(z) = \overline{z} \)

\[
\begin{align*}
U(x, y) &= x & \frac{\partial U}{\partial x} &= 1 & \frac{\partial U}{\partial y} &= 0 \\
V(x, y) &= -y & \frac{\partial V}{\partial x} &= 0 & \frac{\partial V}{\partial y} &= -1
\end{align*}
\]

Cauchy–Riemann equations are not satisfied! \( 1 \neq -1 \).

Conclusion: \( f(z) = \overline{z} \) is not differentiable at any \( z_0 \).

Proof of Theorem 3:

Know that \( \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} \) exists at \( z_0 \).

Let us evaluate this limit in two ways.

\[
\Delta f = f(x_0 + \Delta x, y_0) - f(x_0, y_0)
\]

\[
= U(x_0 + \Delta x, y_0) - U(x_0, y_0) \\
+ (V(x_0 + \Delta x, y_0) - V(x_0, y_0)) i = \Delta U + i \Delta V
\]
\[ \Delta f = \Delta U + i \Delta V \approx \frac{\partial U}{\partial x} \Delta x + i \frac{\partial V}{\partial x} \Delta x \]

\[ \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x} = \lim_{\Delta x \to 0} \left( \frac{\partial U}{\partial x} + i \frac{\partial V}{\partial x} \right) \]

\[ = \frac{\partial U}{\partial x} + i \frac{\partial V}{\partial x} = f'(z_0) \tag{9} \]

On the other hand, we can evaluate the same limit by letting \( \Delta z = i \Delta y \), i.e. approaching \( z_0 \) vertically.

Once again,

\[ \Delta f = \Delta U + i \Delta V \]

\[ \Delta U = U(x_0, y_0 + \Delta y) - U(x_0, y_0) \]

\[ \Delta V = V(x_0, y_0 + \Delta y) - V(x_0, y_0) \]

\[ \Delta U = \frac{\partial U}{\partial y} \Delta y \]

\[ \Delta V = \frac{\partial V}{\partial y} \Delta y \]

\[ f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta U + i \Delta V}{\Delta z} = \lim_{\Delta y \to 0} \frac{\frac{\partial U}{\partial y} \Delta y + \frac{\partial V}{\partial y} \Delta y i}{i \Delta y} \]
\[
\lim_{\Delta y \to 0} \frac{1}{i} \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}
\]

\[
= \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}
\]

Thus we have obtained two expressions for \( f'(z_0) \):

\[
f'(z_0) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \quad i \quad \text{and}
\]

\[
f'(z_0) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}
\]

Evaluating the real and imaginary parts on both sides, we obtain the Cauchy-Riemann equations,

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}
\]
Theorem 4. Let \( f(x+iy) = u(x,y) + iv(x,y)i \). Suppose that \( f(z) \) is defined in a circular neighbourhood of \( z_0 \), \( \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y} \) exist and are continuous near \( z_0 \), and \( \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \) at \( z_0 \). Then \( f(z) \) is differentiable at \( z_0 \).

Example: \( f(z) = e^z \)

\[
e^{x+iy} = e^x \cos(y) + e^x \sin(y)i
\]

\[
\frac{\partial u}{\partial x} = e^x \cos y \quad \frac{\partial v}{\partial x} = e^x \sin y
\]

\[
\frac{\partial u}{\partial y} = -e^x \sin y \quad \frac{\partial v}{\partial y} = e^x \cos y
\]

Conclusion: \( e^z \) is differentiable at every \( z_0 \).