Cauchy's Integral formula.

Let \( \Gamma \) be a positively oriented closed contour, simple, \( f(z) \) is analytic in the interior of \( \Gamma \) and on \( \Gamma \), \( z_0 \) interior point.

Then

\[
2\pi i f(z_0) = \int_{\Gamma} \frac{f(z)}{z-z_0} \, dz
\]

Remarks:
1. \( \int_{\Gamma} \frac{f(z)}{z-z_0} \, dz \neq 0 \) in general, because Cauchy's Integral Theorem does not apply. \( \frac{f(z)}{z-z_0} \) is analytic in \( D = \mathbb{C} \setminus \{ z_0 \} \) which is not simply connected.
2. Substituting \( f(z) = (z-z_0) g(z) \), where \( g(z) \) is analytic, we recover Cauchy's Integral theorem, \( \int_{\Gamma} g(z) \, dz = 0 \).
(3) The values of $f(z)$ on $\Gamma$ determine the values of $f(z_0)$ at any interior point $z_0$.

Proof: Write $\int_{\Gamma} \frac{f(z)}{z-z_0} \, dz$ as $I_1+I_2$,

where $I_1 = \int_{\Gamma} \frac{f(z_0)}{z-z_0} \, dz$ and

$$ I_2 = \int_{\Gamma} \frac{f(z)-f(z_0)}{z-z_0} \, dz. $$

$$ I_1 = f(z_0) \int_{\Gamma} \frac{1}{z-z_0} \, dz $$

$$ = 2\pi i f(z_0). $$

It remains to show that $I_2 = 0$. To do this, we deform $\Gamma$ to $\Gamma_r$.

$$ I_2 = \int_{\Gamma_r} \frac{f(z)-f(z_0)}{z-z_0} \, dz = \int_{\Gamma_r} \frac{1}{z-z_0} \, dz $$

for any $r > 0$. For
\[
\frac{f(z) - f(z_0)}{z - z_0} \text{ has a limit, } f'(z_0), \quad \text{as } z \to z_0.
\]

In particular, it is bounded near \(z_0\), i.e., \(\exists R > 0\) and \(M > 0\) such that

\[
\left| \frac{f(z) - f(z_0)}{z - z_0} \right| < M \iff |z - z_0| < R.
\]

(e.g., can take \(M = |f'(z_0)| + 1\). This means that for any \(r < R\) and such that \(P_r\) lies in interior of \(P\), we have

\[
\left| \int_{P_r} \frac{f(z) - f(z_0)}{z - z_0} \, dz \right| \leq (\text{Length of } P_r) \cdot M \quad \text{(**)}
\]

Now let \(r \to 0\). Since \(\int_{P_r} \frac{f(z) - f(z_0)}{z - z_0} \, dz \)

is independent of \(r\), inequality (**) is satisfied for all \(r \Rightarrow \int_{P} \frac{f(z) - f(z_0)}{z - z_0} \, dz = 0.

Thus \(I_2 = 0\), as claimed.
Examples: (1) \( \Gamma = \text{pos. oriented circle of radius 1 centered at 1} \)

\[
\oint_{\Gamma} \frac{e^{\frac{\pi}{2}iz}}{z^2 - 1} \, dz
\]

Write

\[
\frac{e^{\frac{\pi}{2}iz}}{z^2 - 1}
\]

as

\[
\frac{f(z)}{z - 1}
\]

where \( f(z) = \frac{e^{\frac{\pi}{2}iz}}{z + 1} \)

Apply the integral formula:

\[
\oint_{\Gamma} \frac{e^{\frac{\pi}{2}iz}}{z^2 - 1} \, dz = \oint_{\Gamma} \frac{f(z)}{z - 1} \, dz = 2\pi i \cdot f(1)
\]

\[
= 2\pi i \cdot \frac{e^{\frac{\pi}{2}i}}{2} = \frac{\pi i}{2} \cdot i = -\pi.
\]

(2) \( \oint_{\Gamma_2} \frac{e^{\frac{\pi}{2}iz}}{z^2 - 1} \, dz \), where \( \Gamma_2 = \text{circle of radius 2 centered at 0} \).
\[ \oint_{C_1} \frac{e^{\frac{\pi}{2}i z}}{z^2 - 1} \, dz \]

\[ = \oint_{C_2} \frac{e^{\frac{\pi}{2}i z}}{z^2 - 1} \, dz + \]

By (1), \[ \oint_{C_1} \frac{e^{\frac{\pi}{2}i z}}{z^2 - 1} \, dz = -\pi. \]

To compute second integral, write \[ \frac{e^{\frac{\pi}{2}i z}}{z^2 - 1} \] as \[ \frac{g(z)}{z + 1}, \] where \[ g(z) = \frac{e^{\frac{\pi}{2}i z}}{z - 1}. \]

By integral formula, \[ \oint_{C_2} \frac{g(z)}{z + 1} \, dz = 2\pi i \cdot g(-1) \]

\[ C_2 = 2\pi i \cdot \frac{-i}{-2} = -\pi \]

Answer: \[ -\pi + (-\pi) = -2\pi. \]
Write integral formula as
\[ f(z) = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(w)}{w-z} \, dw \]
and think of LHS as a function of z. Can differentiate with respect to z under the integral sign. More generally, suppose \( \Gamma \) is any contour (not nec. closed), \( f(z) \) is a continuous function on \( \Gamma \). Set
\[ G(z) = \oint_{\Gamma} \frac{f(w)}{w-z} \, dw \]
for any \( z \), not on \( \Gamma \). Then \( G(z) \) is analytic in \( \mathbb{C} \setminus \Gamma \), and
\[ G'(z) = \oint_{\Gamma} \frac{f(w)}{(w-z)^2} \, dw \]
Proof analyzes the integral quotient
\[ \frac{G(z + \Delta z) - G(z)}{\Delta z} \]
as \( \Delta z \to 0 \)
see book for details.
Applying this to the case where \( \Gamma \) is a simple loop in \( C \), \( f(w) \) is analytic on and inside of \( \Gamma \), we obtain

\[
f'(z) = \frac{1}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w-z)^2} \, dw = \frac{1}{2\pi i} \int_{\Gamma} f(w)(w-z)^{-2} \, dw
\]

Can differentiate with respect to \( z \) under the integral sign again:

\[
f''(z) = \frac{1}{2\pi i} \int_{\Gamma} f(w)(-2)(w-z)^{-3} \, dw
\]

\[
= \frac{2}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w-z)^3} \, dw
\]

One more time:

\[
f'''(z) = \frac{2 \cdot 3}{2\pi i} \int_{\Gamma} \frac{f(w)}{(w-z)^4} \, dw
\]
Generalized Cauchy Integral

Formula

If \( \Gamma \) is a simple closed contour, \( f(z) \) is analytic on and inside \( \Gamma \). Then for any \( n \neq 0 \) and any \( z \) in interior of \( \Gamma \), the \( n \)-th derivative of \( f(z) \),

\[
f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\gamma} \frac{f(w)}{(w-z)^{n+1}} \, dw
\]

Corollary: If \( f(z) \) is analytic at \( z_0 \), then \( f'(z_0), f''(z_0), f'''(z_0), \ldots, f^{(n)}(z_0), \ldots \) exist. That is \( f(z) \) is differentiable to any order at \( z_0 \).

Proof of Corollary: If \( f(z) \) is differentiable in a disc of radius \( r \) centered at \( z_0 \), take \( \Gamma = \text{pos. oriented circle of radius } r/2 \) centered at \( z_0 \).
Apply the generalized Cauchy Integral Formula to this contour, with \( z = z_0 \).

A.E.D.

**Examples:** \( \Gamma \) = positively oriented unit circle.

1. \( \int_{\Gamma} \frac{\cos(z)}{z} \, dz = 2\pi i \cos(0) = 2\pi i \)

2. \( \int_{\Gamma} \frac{\sin(3z)}{z^2} \, dz = \left( \frac{1}{2\pi i} \right) \left( \frac{1}{3} \right) f'(0) \), where
   
   \( f(z) = \sin(3z) \), \( f'(z) = \cos(3z) \cdot 3 \)

   \( f'(0) = 3 \). Thus \( \int_{\Gamma} \frac{\sin(3z)}{z^2} \, dz = 2\pi i \cdot 3 \)

   \( = 6\pi i \)
(3) $\int_{\Gamma} \frac{e^{z} + z}{(z-2)^3} \, dz = 0$

by Cauchy's Theorem.

(4) Let $C$ be the pos. oriented circle of radius 3 centered at 0. Find

$$\int_{C} \frac{e^{z} + z}{(z-2)^3} \, dz .$$

Rewrite integral as $\int_{C} \frac{f(z)}{(z-2)^3} \, dz$, where $f(z) = e^{z} + z$.

By Integral Formula (with $n=2$),

$$\int_{C} \frac{f(z)}{(z-2)^3} \, dz = \frac{2\pi i}{2!} \cdot f''(2) = \frac{2\pi i}{2} \cdot e^2 = \pi e^2 i .$$

$f'(z) = e^z + 1$, $f''(z) = e^z$