Last time: Contour integrals.

Defined

\[ \oint_{\Gamma} f(z) \, dz \]

If \( \Gamma \) is smooth, parametrized by \( z(t), \ a \leq t \leq b \), then

\[ \oint_{\Gamma} f(z) \, dz = \int_{a}^{b} f(z(t)) \, z'(t) \, dt \]

**Example:** \( f(z) = z^n \), \( \Gamma \) = pos. oriented unit circle.

\( z(t) = e^{it}, \ 0 \leq t \leq 2\pi \)

\[ \oint_{\Gamma} f(z) \, dz = \oint_{\Gamma} z^n \, dz = \left\{ \begin{array}{ll}
0, & \text{if } n \neq -1 \\
2\pi i, & \text{if } n = -1
\end{array} \right. \]
Another example: \[ \oint_{\Gamma} \overline{z} \, dz \]

\[ \Gamma = \text{pos. oriented unit square} \]

\[ \Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 \]

\[ \Gamma_1: \quad z = t, \quad 0 \leq t \leq 1 \]
\[ z'(t) = 1 \]

\[ \oint_{\Gamma_1} \overline{z} \, dz = \int_0^1 t \cdot 1 \, dt = \frac{t^2}{2} \bigg|_0^1 = \frac{1}{2} \]

\[ \Gamma_2: \quad z(t) = 1 + ti, \quad 0 \leq t \leq 1 \]
\[ z'(t) = i \]

\[ \oint_{\Gamma_2} \overline{z} \, dz = \int_0^1 (1-t)i \cdot i \, dt = \int_0^1 i + t \, dt = i + \frac{1}{2} \]

\[ \Gamma_3: \quad z(t) = (1-t) + i, \quad z'(t) = -1, \quad \overline{z}(t) = (1-t)-i \]

\[ \oint_{\Gamma_3} \overline{z} \, dz = \int_0^1 (1-t-i)(-1) \, dt = i - 1 + \int_0^1 t \, dt = i - \frac{1}{2} \]

\[ \Gamma_4: \quad z = 1 + i \]

\[ \oint_{\Gamma_4} \overline{z} \, dz = 0 \]
\[ \Gamma_4: \quad z(t) = (1-t)i, \quad 0 \leq t \leq 1 \]
\[ \bar{z}(t) = (t-1)i, \quad z'(t) = -i \]
\[ \int \bar{z} \, dz = \int_0^1 (t-1)i \, (-i) \, dt = \int_0^1 (t-1) \, dt \]
\[ = \frac{1}{2} - 1 = -\frac{1}{2}. \]

\[ \int \bar{z} \, dz = \int \bar{z} \, dz + \int \bar{z} \, dz + \int \bar{z} \, dz + \int \bar{z} \, dz \]
\[ \Gamma = \Gamma_1 + \Gamma_2 + \Gamma_3 + \Gamma_4 \]
\[ = \frac{1}{2} + (1 + \frac{1}{2}) + (1 - \frac{1}{2}) + (-\frac{1}{2}) = 2i \]

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**Theorem:** Suppose \( F'(z) = f(z) \) for any \( z \) in a domain \( D \). If \( \Gamma \subset D \) is contour with initial point \( z_0 \) and terminal point \( z_1 \), then
\[ \int_{\Gamma} f(z) \, dz = F(z_1) - F(z_0). \]

In particular, if \( \Gamma \) is closed, then
\[ \int_{\Gamma} f(z) \, dz = 0. \]
Examples.

(1) \[ \int_{\Gamma} z^2 \, dz = F(i) - F(0) \]
\[ \Gamma = -\frac{1}{3} i \]
Here \( F'(z) = \frac{1}{3} z^3 \)
\[ F'(z) = z^2 \]

(2) \[ \int_{\Gamma} \sin(z) \, dz \], same \( \Gamma \).
Here \( F'(z) = -\cos(z) \), \( F'(z) = \sin(z) \).
\[ \int_{\Gamma} \sin(z) \, dz = F(i) - F(0) = -\cos(i) + 1 \]
\[ = -\frac{e^{i}i + e^{-i} \cdot i}{2} + 1 = -\frac{e^{-1} + e}{2} + 1 \]

To prove Theorem, need

Lemma: Suppose \( F(z) \) is differentiable as a function of \( z \), \( z(t) \) is differentiable as a function of \( t \). Then
\[ \frac{d}{dt} F(z(t)) = F'(z(t)) \cdot z'(t) \]
Proof of Lemma:

Write $z(t) = x(t) + i y(t)$

$f(z) = u(x,y) + i v(x,y)$.

$$\frac{d}{dt} f(z(t)) = \frac{d}{dt} \left( u(x(t), y(t)) + i v(x(t), y(t)) \right) = \frac{d}{dt} \left( u(x(t), y(t)) \right) + \frac{d}{dt} \left( v(x(t), y(t)) \right) = \frac{\partial u}{\partial x} x'(t) + \frac{\partial v}{\partial x} y'(t) + \frac{\partial u}{\partial y} y'(t) + \frac{\partial v}{\partial y} x'(t) = \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) (x'(t) + i y'(t)) = \left( \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \right) (x'(t) + i y'(t)) \quad (\text{C. R. equations})$$

$$f'(z(t)) \cdot z'(t), \text{ as claimed.}$$
Proof of Theorem: May assume $\Gamma$ is smooth.

\[
\int_{\Gamma} f'(z) \, dz = \int_{\Gamma_1} f'(z) \, dz + \int_{\Gamma_2} f'(z) \, dz.
\]

\[
= (F'(w) - F(z_0)) + (F(z_1) - F(w)) \quad \text{Assuming Theorem holds for } \Gamma_1 \text{ and } \Gamma_2.
\]

Now let us assume $\Gamma$ is smooth. Parametrize by $z(t)$, $a \leq t \leq b$.

Then
\[
\int_{\Gamma} f(z) \, dz = \int_{a}^{b} f(z(t)) z'(t) \, dt
\]

\[
= \int_{a}^{b} F'(z(t)) z'(t) \, dt
\]
\[
= \int_{a}^{b} \left( \frac{d}{dt} F(z(t)) \right) \, dt
\]

\[
= \left. F(z(t)) \right|_{t=a}^{t=b} = F(z(b)) - F(z(a))
\]

\[
= F(\text{terminal pt.}) - F(\text{initial pt.})
\]
Corollary: There does not exist an analytic function in the punctured disk \( D_R \) of radius \( R \) centered at 0 whose derivative is \( \frac{1}{z} \).

\( D_R = \{ z \in \mathbb{C} \mid |z| < R, z \neq 0 \} \)

Here \( R \) is any positive real number.

Proof: Enough to show that

\[ \int_{\Gamma_r} \frac{1}{z} \, dz \neq 0 \text{ for any } r > 0. \]

\( \Gamma_r \) is the positively oriented circle of rad. \( r \) centered at 0.

Parametrize \( \Gamma_r \): \( z(t) = re^{it}, 0 \leq t \leq 2\pi \)

\[ \int_{\Gamma_r} \frac{1}{z} \, dz = \int_0^{2\pi} \frac{1}{re^{it}} \cdot r \cdot ie^{it} \, dt \]

\[ = \int_0^{2\pi} i \, dt = 2\pi i. \]
Theorem: Let \( f(z) \) be a continuous function in a domain \( D \). Then the following conditions are equivalent.

1. \( f(z) \) has an anti-derivative in \( D \).
2. \( \int_{\Gamma} f(z)\,dz = 0 \) for any closed contour \( \Gamma \) in \( D \).
3. \( \int_{\Gamma} f(z)\,dz \) depends only on the endpoints of \( \Gamma \). Here \( \Gamma \) is not assumed to be closed.

(3) is called "independence of path".

Proof: (1) \( \implies \) (2). Follows from previous theorem.

(2) \( \implies \) (3). Suppose \( \Gamma_1 \) and \( \Gamma_2 \) both connect \( z_0 \) to \( w_0 \).
Consider the closed contour $\Gamma$ formed by $\Gamma_1 \cup (-\Gamma_2)$. Here $-\Gamma_2$ means $\Gamma_2$ with opposite orientation.

By (2) \( \int_{\Gamma} f(z)\,dz = 0 \). Thus

\[ \int_{\Gamma_1} f(z)\,dz - \int_{\Gamma_2} f(z)\,dz = 0 \] or

\[ \int_{\Gamma_1} f(z)\,dz = \int_{\Gamma_2} f(z)\,dz. \]

Independence of path!
Remains to prove \((3) \Rightarrow (1)\).
Assume \((3)\). Want to construct \(F(z) = \text{anti-derivative of } f(z)\) in \(D\).
Fix a point \(z_0 \in D\). Set
\[
\int_{\Gamma} f(w) \, dw = F(z)
\]
where \(\Gamma\) is a contour with initial point \(z_0\) and terminal point \(z\). Note that \(F(z)\) is independent of our choice of \(\Gamma\) as long as \(\Gamma \subset D\).

**Claim:** \(F'(z) = f(z)\) for any \(z \in D\).

\[
F(z + \Delta z) - F(z) = \int_{\Gamma} f(z) \, dz - \int_{\Gamma'} f(z) \, dz
\]
\[
= \int_{\Gamma} f(z) \, dz
\]
where \( f \) = directed interval with initial point \( z \) and terminal point \( z + \Delta z \).

Parametrize \( f \phi(t) = z + t \Delta z \), \( 0 \leq t \leq 1 \).

Thus \( F(z + \Delta z) - F(z) = \int \phi^{'}(t) \Delta z \, dt \)

\[
\frac{F(z + \Delta z) - F(z)}{\Delta z} = \int_{0}^{1} f(z + t\Delta z) \, dt
\]

As \( \Delta z \to 0 \), this integral converges to \( f(z) \). (Check!)}

**Conclusion:**

\[
F^{'}(z) = f(z) \quad \text{Q.E.D.}
\]