Elementary functions
- Polynomials
- Rational functions
- Exponential
- Trig. functions

Extend each type of function from \( \mathbb{R} \) to \( \mathbb{C} \).

Today: Logarithmic + power functions, inverse trig. functions

Logarithmic Function

Inverse to exponential function

\[ z \rightarrow e^z \]

\[ \log(z) = w \text{ means } e^w = z. \]

Suppose \( z = r e^{i\theta} \) is given.

Want to solve \( e^w = z \) for \( w \).
Write $w = a + bi$, $a, b - \text{real}$.

$$e^w = e^a \cdot e^{bi} = z = re^{i\theta} \quad r = |z|$$

polar form of $e^w$  

polar form of $z$  

Thus $r = e^a$

$$b = \theta + 2\pi k, \quad k \in \mathbb{Z}$$

$$\begin{cases} a = \ln(r) \quad \text{Here } \ln(r) \text{ is the} \\ b = \theta + 2\pi k \quad \text{natural log of} \\
\end{cases}$$

the positive real no. $r$.

$$\begin{cases} a = \ln |z| \\ b = \arg(z) - \text{multi-valued} \end{cases}$$

$\log(z)$ denotes the set of all values

$\log(z)$ principal value, with $\arg(z)$ replaced by $\text{Arg}(z) \in (-\pi, \pi]$

$$\log(z_1 z_2) = \log(z_1) + \log(z_2)$$

$$\log\left(\frac{z_1}{z_2}\right) = \log(z_1) - \log(z_2)$$
These equalities fail if we replace \( \log \) by \( \Log \).

**Theorem:** (a) \( \Log(z) \) is analytic in the domain \( D = \mathbb{C} \setminus \{ \text{non-positive real axis} \} \).

(b) \( \frac{d}{dz} \Log(z) = \frac{1}{z} \) for every \( z \) in \( D \).

**Proof:** (a) Check that \( f(z) = \Log(z) \) satisfies the Cauchy-Riemann equations for every \( z \in D \). Will use the C-R. equations in polar form (from HW problem).

\[
\begin{align*}
  f(z) &= u(r, \Theta) + v(r, \Theta) i \\
  \frac{\partial u}{\partial r} &= \frac{1}{r} \frac{\partial v}{\partial \Theta}, \quad \frac{\partial v}{\partial r} = -\frac{1}{r} \frac{\partial u}{\partial \Theta} \\
  r &= |z|, \quad \Theta = \text{Arg}(z)
\end{align*}
\]
In our case \( f(z) = \log(z) = \frac{\ln(r) + \Theta i}{\sqrt{r}} \)

\[
\frac{\partial u}{\partial r} = \frac{1}{r}, \quad \frac{\partial u}{\partial \Theta} = 0
\]

\[
\frac{\partial v}{\partial r} = 0, \quad \frac{\partial v}{\partial \Theta} = 1.
\]

The C-R equations are satisfied.

Thus \( \log(z) \) is differentiable at every point of \( D \).

(b) \( e^{\log(z)} = z \). Differentiate both sides:

\[
e^{\log(z)} \cdot \frac{d}{dz} \log(z) = 1
\]

\[
z \cdot \frac{d}{dz} \log(z) = 1
\]

\[
\frac{d}{dz} \log(z) = \frac{1}{z}, \text{ as claimed.}
\]
Corollary: a) $z \to \ln |z|$ is harmonic in the plane, with 0 removed.

(b) $z \to \text{Arg}(z)$ is harmonic in $D = \mathbb{C} \setminus \text{(non-positive real axis)}$.

The function in (a) can be rewritten as $f(x,y) = \ln \sqrt{x^2+y^2} = \frac{1}{2} \ln(x^2+y^2)$

Example: Where is the function $f(z) = \log(4z^2-y)$ differentiable, and what is the derivative $f'(z)$?

Solution: Need to exclude $z$ values, where $4z^2-y$ is a non-positive real no.

Write $4z^2-y = -x$, where $x \geq 0$.

$z^2 = \frac{4-x}{4}$

Case 1: $x \in [0,4]$. Then $z$ is a real number between $-1$ and 1.
Case 2: $x \geq y$. Here $z$ is any purely imaginary number.

Thus need to exclude the $y$-axis and the interval $[-1, 1]$ on $x$-axis.

$f(z)$ is analytic in $D = \mathbb{C} \setminus \left( \text{Y-axis, interval } [-1, 1] \text{ on x-axis} \right)$.

For $z$ in $D$, $f'(z)$ can be computed by chain rule:

$$f'(z) = \frac{1}{4z^2 - 4} \cdot 8z = \frac{2z}{z^2 - 1}.$$

Branch chasing
Another example:

Compare \( \log(4z^2-4) \) with \( \log(2z+2) + \log(2z-2) \).

\( \log(4z^2-4) \) is differentiable in \( \mathbb{C} \setminus \left\{ \text{y-axis interval} \left\{ \left[ -1, 1 \right] \right\} \text{on x-axis} \right\} \).

\( \log(2z+2) \) is differentiable in \( \mathbb{C} \setminus \left\{ \text{interval} \left( -\infty, -1 \right] \right\} \text{on x-axis} \).

\( \log(2z-2) \) is differentiable in \( \mathbb{C} \setminus \left\{ \text{interval} \left( -\infty, 1 \right] \right\} \text{on x-axis} \).
\[ \log(4z^2 - 4) \quad \log(2z+2) + \log(2z-2) \]

The complex power function

\( f(z) = z^\alpha \). Know what \( z^\alpha \) is when \( \alpha \) is an integer. Also considered the case, where \( \alpha = \frac{1}{n} \) (\( n \) th root of \( z \)).

What if \( \alpha \) is an arbitrary complex number? For example, what is \( i \)?

Write \( z^\alpha = e^{\log(z^\alpha)} = e^{\alpha \log(z)} \).

Multi-valued in general, because \( \log(z) \) is multi-valued.
Example: What is $i^i$?

\[ i^i = e^{i \log(i)} \]

\[ i = 1 \cdot e^{\frac{\pi i}{2}} \cdot \log(i) = \ln(1) + \frac{\pi}{2} i + 2\pi ki \]

\[ i^i = e^{i \log(i)} = e^{\left(-\frac{\pi i}{2} + 2\pi ki\right)} \]

**Surprise:** all values of $i^i$ are real and positive!

Principal value of $i^i$ is obtained by choosing the value $\log(i) = \frac{\pi i}{2}$ among the infinitely many values of $\log(i)$. Thus \[ i^i = e^{-\pi/2} \]

If $\lambda = \frac{1}{n}$, we recover the formula for $n$th roots of $z = re^{i\theta}$. Here \[ \log(z) = \ln(r) + 2\pi i k, \quad k \text{-integer} \]
\[ z^{\frac{1}{n}} = e^{\frac{1}{n} \log(z)} = e^{\frac{1}{n} \left( \ln(r) + \Theta i + 2\pi k i \right)} \\
= e^{\frac{1}{n} \ln(r)} e^{\frac{1}{n} (\Theta + 2\pi k) i} \\
= r^{\frac{1}{n}} e^{\frac{1}{n} (\Theta + 2\pi k) i}, \quad k \text{-integer.} \\
\]  

\[ n \text{ distinct values for } k = 0, 1, 2, \ldots, n-1. \]

**Theorem:** Let \( z \neq 0 \) be a complex number.

Then \( z^x \) assumes

- one value if \( x \) is a (real) integer
- finitely many values if \( x \) is a (real) rational number
- infinitely many values in all other cases.