5.1.15. First assume that $L < 1$. The argument outlined in the hint shows that there exists an integer $J > 0$ and a real number $0 < m < 1$ such that $c_{j+1}/c_j < m$ for all $j \geq J$. Now let

$$M_k = |c_j|m^{k-J} = \frac{|c_j|}{m^J}m^k.$$  

Then for any $j \geq J$,

$$|c_j| \leq M_j$$  

(see the hint). Since

$$\sum_{j=0}^{\infty} M_j = \frac{c_j}{m^J} \sum_{k=0}^{\infty} m^k$$

converges (it is a geometric series), the series $\sum_{j=0}^{\infty} c_j$ also converges by the comparison test.

Now assume that $L > 1$. In this case there exists an integer $J \geq 0$ such that $|c_{j+1}|/c_j > 1$ for any $j \geq J$. That is,

$$|c_{j+1}| > |c_j| > |c_{j-1}| > \ldots > |c_J|.$$  

This tells us that

$$\lim_{n \to \infty} |c_j| \neq 0.$$  

Thus the series $\sum_{j=0}^{\infty} c_j$ diverges by the $n$th term test.

5.2.4. $f(z) = e^{\alpha \Log(1+z)}$ is analytic in $|z| < 1$. Moreover, $f(0) = 1$,

$$f^{(j)}(z) = \alpha(\alpha - 1) \cdots (\alpha - j + 1)e^{(\alpha-j)\Log(1+z)}$$  

$$f^{(j)}(0) = \alpha(\alpha - 1) \cdots (\alpha - j + 1),$$  

and the claim follows.

5.2.14. Suppose the Taylor series for $f(z)$ in $D$ is

$$f(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \ldots$$

Then $a_k = \frac{f^{(k)}(z_0)}{k!} = 0$ for $k = 0, 1, 2, \ldots$. Thus $f(z)$ is identically zero in $D$. 
5.2.15. Arguing as in the previous exercise (with $z_0 = 0$), we see that

$$f(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \ldots$$

and

$$a_k = \frac{f^{(k)}(z_0)}{k!} = 0$$

whenever $k$ is odd. Thus

$$f(z) = a_0 + a_2 z^2 + a_4 z^4 + a_6 z^6 + \ldots$$

and consequently, $f(-z) = f(z)$.

5.2.16. Note a typo in the statement of the problem: $p_n(z)$ is meant to be $p(z)$. Note also that

$$p(z) = c_0 + c_1 (z - 1) + \cdots + c_n (z - 1)^n$$

is a Taylor series for $p(z)$ at $z = 1$. Thus $c_j = \frac{p^{(j)}(1)}{j!}$. In particular,

$$c_0 = \frac{p(1)}{0!} = \sum_{j=0}^{n} a_j$$

$$c_1 = \frac{p'(1)}{1!} = a_1 + 2a_2 + \ldots + na_n,$$

$$c_2 = \frac{p''(1)}{2!} = 2 \cdot 1a_2 + 3 \cdot 2a_3 + \cdots + n(n-1)a_n,$$

etc. In general,

$$p^{(j)}(z) = j \cdot (j - 1) \cdots \cdot 2 \cdot 1 + (j + 1) \cdot j \cdots \cdot 2z + \cdots + n(n-1) \cdots (n-j+1)z^{n-j}$$

and thus

$$c_j = \frac{p^{(j)}(1)}{j!} = \frac{1}{j!} \cdot \left( j \cdot (j - 1) \cdots \cdot 2 \cdot 1 + (j + 1) \cdot j \cdots \cdot 2 + \cdots + n(n-1) \cdots (n-j+1) \right).$$

5.2.18. (a)

$$\left| e^z - \sum_{k=0}^{n} \frac{z^k}{k!} \right| = \left| \sum_{k=n+1}^{\infty} \frac{z^k}{k!} \right| \leq \sum_{k=n+1}^{\infty} \left| \frac{z^k}{k!} \right| \leq \sum_{k=n+1}^{\infty} \frac{1}{k!} = \sum_{j=0}^{\infty} \frac{1}{(n+j+1)!} = \sum_{j=0}^{\infty} \frac{1}{(n+j+1)!} \cdot \left( \frac{1}{n+2} \right)^j = \frac{1}{(n+1)!} \sum_{j=0}^{\infty} \left( \frac{1}{n+2} \right)^j.$$

Here $j = k - n - 1$. Using the formula for $\sum_{j=0}^{\infty} c^j = \frac{1}{1-c}$ with $c = \frac{1}{n+2}$, we see that

$$\sum_{j=0}^{\infty} \left( \frac{1}{n+2} \right)^j = \left( \frac{1}{1 - \frac{1}{n+2}} \right) = \frac{n+2}{n+1} = 1 + \frac{1}{n+1},$$

and part (a) follows.
(b) \[
\left| \sin(z) - \sum_{k=0}^{n} (-1)^{k} \frac{z^{2k+1}}{(2k+1)!} \right| = \sum_{k=n+1}^{\infty} \frac{(-1)^{k} z^{2k+1}}{(2k+1)!} = \\
\sum_{k=n+1}^{\infty} \left| (-1)^{k} \frac{z^{2(k+1)}}{(2k+1)!} \right| = \sum_{k=n+1}^{\infty} \frac{1}{(2k+1)!}. 
\]
Setting \( j = k - n - 1 \), we obtain
\[
\sum_{k=n+1}^{\infty} \frac{1}{(2k+1)!} = \sum_{j=0}^{\infty} \frac{1}{(2n + 4j + 3)!} \left( 1 + c + c^2 + c^3 + \ldots \right) \leq \frac{1}{(2n + 3)!} \left( 1 + \frac{1}{1 + c} c + c^2 + c^3 + \ldots \right)
\]
where \( c = \frac{1}{(2n + 4)(2n + 5)} \). Once again, using the formula \( \sum_{j=0}^{\infty} c^j = \frac{1}{1-c} \) for the sum of the geometric series, we see that
\[
1 + c + c^2 + c^3 + \ldots = \frac{1}{1 - \frac{1}{(2n + 4)(2n + 5)}} = \frac{(2n + 4)(2n + 5)}{(2n + 4)(2n + 5) - 1} = \frac{4n^2 + 18n + 20}{4n^2 + 18n + 19},
\]
as desired.

5.3.2. \[
\lim_{j \to \infty} \left| \frac{a_{j+1}(z - z_0)^{j+1}}{a_j(z - z_0)^j} \right| = \lim_{j \to \infty} \left| \frac{a_{j+1}}{a_j} \right| \left| z = z_0 \right| = L |z - z_0|.
\]
By the ratio test (Theorem 2 on page 237), the series is convergent if \( L |z - z_0| < 1 \) (i.e. if \( |z - z_0| < 1/L \)) and diverges if \( L |z - z_0| > 1 \) (i.e. if \( |z - z_0| > 1/L \)). This shows that \( R = 1/L \) is the radius of convergence.

5.3.4. No. Indeed, suppose the series converges at \( 2 + 3i \). Then the radius of convergence \( R \) is \( \geq |2 + 3i - 0| = \sqrt{13} \). On the other hand, the distance from the origin to \( 3 - i \) is \( \sqrt{10} \), which is less than \( \sqrt{13} \). This means that \( 3 - i \) is within the disk of convergence, so the series converges at \( 3 - i \).

5.3.6. (a) \( f(0) = 1 \) agrees with the series at \( z = 0 \). For \( z \neq 0 \) divide the Maclaurin expansion
\[
\sin(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots
\]
by \( z \) to obtain the desired Taylor expansion for \( f(z) \).

(b) The series for \( f(z) \) in part (a) converges for every complex number \( z \). This follows from the fact that the Maclaurin series for \( \sin(z) \) converges for every \( z \) (or, alternatively, from the ratio test). By Theorem 10, \( f(z) \) is analytic in the entire plane.

(c) \( f^{(3)}(0) = 3! a_3 \), where \( a_3 \) is the coefficient of \( z^3 \) in the power series for \( f(z) \) in part (a). Since this series has no \( z^3 \) term, \( a_3 = 0 \), and thus \( f^{(3)}(0) = 0 \) as well.
Similarly, \( f^{(4)}(0) = 4!a_4 = 4! \frac{1}{5!} = \frac{1}{5} \).

5.3.12. Substituting \( f(z) = \sum_{k=0}^{\infty} a_k z^k \) into the equation, we see that \( a_0 = f(0) = 1 \) and \( ka_k = a_{k-1} \) for all \( k \geq 1 \). Thus \( a_1 = 1, a_2 = 1/2, a_3 = 1/3a_2 = 1/3! \), etc. Continuing this way we see that \( a_k = 1/k! \) for every \( k \geq 0 \) and thus \( f(z) = \sum_{k=0}^{\infty} \frac{1}{k!} z^k = e^z \).