4.4.10. The domains of analyticity are:
(a) \( \mathbb{C}\{5i, -5i\} \),
(b) \( \mathbb{C} \),
(c) \( \mathbb{C}\{3 - i, 3 + i\} \),
(d) \( \mathbb{C}\{z; \text{Re}(z) \leq -3, \text{Im}(z) = 0\} \),
(e) \( \mathbb{C}\{\pi + 2\pi n | n \in \mathbb{Z}\} \).

In each case \( f \) is analytic in the open disk \( |z| < 2.5 \). This disk is a simply connected domain containing the circle \( |z| = 2 \). The integral is thus 0 by Cauchy’s Integral Theorem (Theorem 3).

4.4.16.
\[
\int_{|z|=1} f(z)dz = \int_{|z|=1} \frac{A_k}{z^k}dz + \int_{|z|=1} \frac{A_{k-1}}{z^{k-1}}dz + \ldots + \int_{|z|=1} \frac{A_1}{z}dz + \int_{|z|=1} g(z)dz = 0 + 0 + \ldots + 2\pi i A_1 + 0 = 2\pi i A_1.
\]
Here I am using Example 2 in Section 4.2 (with \( z_0 = 0 \)) to evaluate the first \( k \) integrals. The last integral = 0 by Cauchy’s theorem.

4.5.4. In parts (a) and (b), we use Cauchy’s integral formula (Theorem 19).
(a) Here we set \( f(z) = \frac{z + i}{z + 2} \), \( n = 1 \) and \( z_0 = 0 \):
\[
\int_C \frac{z + i}{z^3 + 2z^2}dz = 2\pi i f'(0) = \pi/2 + \pi i.
\]
(b) Here we set \( f(z) = \frac{z + i}{z + 2} \), \( n = 0 \) and \( z_0 = -2 \):
\[
\int_C \frac{z + i}{z^3 + 2z^2}dz = 2\pi i f(-2) = -\pi/2 - \pi i.
\]
(c) The function \( \frac{z + i}{z^3 + 2z^2} \) is analytic in the (simply connected) open disk \(|z - 2i| < 2\) containing \( C \). Thus
\[
\int_C \frac{z + i}{z^3 + 2z^2} \, dz = 0
\]
by Cauchy’s integral theorem.

4.5.6. Let \( \Gamma_1 \) be the half circle from 3 to \(-3\) on the upper half plane and \( \Gamma_2 \) be the half circle from \(-3\) to 3 on the lower half plane such that \( \Gamma = \Gamma_1 + \Gamma_2 \). Let \( L \) be the line from \(-3\) to 3. Then
\[
\int_{\Gamma} e^{iz}/(z^2 + 1) \, dz = \int_{\Gamma_1 + L} e^{iz}/(z^2 + 1) \, dz + \int_{-L + \Gamma_2} e^{iz}/(z^2 + 1) \, dz = \int_{\Gamma_1 + L} e^{iz}/(z - i)^2 \, dz + \int_{-L + \Gamma_2} e^{iz}/(z + i)^2 \, dz = 2\pi i(-e^{-1}/2 + 2\pi i(0)) = \pi/e.
\]

4.5.15. \( F(z) \) is continuous in \(|z| \leq 1\). It is immediate from the definition of \( F(z) \) as the ratio of two analytic functions that \( F(z) \) is analytic at any point of the disk \(|z| \leq 1\), as long as \( z \neq 0 \). Our goal is thus to show that \( F(z) \) is analytic at 0.

Since \( f(z)/z \) is continuous on the unit circle, Theorem 15, tells us that \( G(z) \) is analytic in the open unit disk, \(|z| < 1\). It thus suffices to show that \( F(z) = G(z) \) for any \( z \) such that \(|z| < 1\).

First consider the case, where \( z = 0 \). Here
\[
G(0) = \frac{1}{2\pi i} \int_{|z|=1} \frac{f(\zeta)}{\zeta^2} \, d\zeta = f'(0) = F(0),
\]
as desired.

Now assume \( z \neq 0 \) and \(|z| < 1\). Here
\[
G(z) = \frac{1}{z} \left( \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z} \, d\zeta - \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta} \, d\zeta \right) \text{ (partial fractions)}
\]
\[
= \frac{1}{z} (f(z) - f(0)) = F(z)
\]
Therefore \( F(z) = G(z) \) is analytic in the open unit disk, \(|z| < 1\), as claimed.
4.6.4. From Theorem 20 it follows that
\[ |p^{(k)}(0)| = |k! a_k| \leq k! M/1. \]
Thus \( |a_k| \leq M \).

4.6.8. The function \( f(z)/3z^2 \) is analytic in the annulus \( 1 \leq |z| \leq 2 \). Moreover, \( |f(z)/3z^2| \leq 1 \) at the boundary of this annulus, where \( |z| = 1 \) or \( |z| = 2 \). By Theorem 24, \( |f(z)/3z^2| \) attains its maximal value on the boundary of the annulus. That is, \( |f(z)/3z^2| \leq 1 \) for any \( z \) such that \( 1 \leq |z| \leq 2 \). In other words, \( |f(z)| \leq 3|z|^2 \) for any \( z \) such that \( 1 \leq |z| \leq 2 \).

4.6.14. If \( f \) is non-zero then \( g(z) = \frac{1}{f(z)} \) is analytic in \( D \). By Theorem 23 \( |g(z)| = \frac{1}{|f(z)|} \) attains its maximum value on the boundary of \( D \). Thus \( f(z) \) attains its minimum value on the boundary of \( D \).

If \( f(z) \) is allowed to vanish in \( D \), then the minimum modulus principle is no longer valid. For example, consider the function \( f(z) = z \) and \( D = \{|z| \leq 1\} \) is the unit disk. In this case \( |f| \) attains its minimal value at \( z = 0 \) which is not a boundary point of \( D \).

5.1.8. (a) We know that \( \sum_{j=0}^{\infty} c_j \) converges. This means that for any \( \epsilon > 0 \) there exists an \( N \) such that
\[ \left| \sum_{j=0}^{n} c_j - S \right| < \epsilon, \text{ for all } n \geq N \]
Then \( \left| \sum_{j=0}^{n} c_j - S \right| = \left| \sum_{j=0}^{n} c_j - S \right| < \epsilon, \text{ for all } n \geq N \)
and so \( \sum_{j=0}^{n} c_j \) converges to \( S \).
(b) and (c) are similar to (a).

5.1.14. (a) Note that \( |j+i| = \sqrt{j^2+1} > j \) for every positive integer \( j \). Thus
\[ \left| \frac{1}{j(j+i)} \right| = \frac{1}{j \cdot |j+i|} < \frac{1}{j^2}. \]
Since \( \sum_{j=1}^{\infty} 1/j^2 \) converges, so does \( \sum_{j=1}^{\infty} \frac{1}{j(j+i)} \) by the comparison test (Theorem 1).
(b) Use the comparison test with $M_k = \frac{1}{k^{3/2}}$.

(c) Use the comparison test with $M_n = \frac{1}{k^2}$.

(d) Use the comparison test with $M_k = \frac{12}{k^2}$. To see that the comparison test applies, note that for every $k \geq 8$ we have $5k + 8 \leq 5k + k = 6k$ and $k^3/2 > 1$ and thus $k^3 - 1 > k^3 - k^3/2 = k^3/2$. Thus

$$\left|(-1)^k \left(\frac{5k + 8}{k^3 - 1}\right)\right| = \frac{5k + 8}{k^3 - 1} < \frac{6k}{k^3/2} = \frac{12}{k^2}.$$ 

5.1.20.

$$\left|\sum_{j=0}^{n} z^j - \frac{1}{1 - z}\right| = \left|\frac{z^{n+1}}{1 - z}\right|$$

So for any $\epsilon > 0$ we can find a $z$ close enough by 1 such that

$$\left|\frac{z^{n+1}}{1 - z}\right| > \epsilon$$

in other words, there is no $N$ so that $n > N$ implies

$$\left|\frac{z^{n+1}}{1 - z}\right| < \epsilon \quad \text{for all } z,$$

and hence the series does not converge uniformly.