2.5.2. We begin by computing the partial derivatives of $f(x, y) = ax^2 + bxy + cy^2$:

$$f_x = 2ax + by, \quad f_{xx} = 2a, \quad f_y = bx + 2cy, \quad f_{yy} = 2c.$$  

The Laplace equation, $f_{xx} + f_{yy} = 0$ holds if and only if $-2a = 2c$ or equivalently, $c = -a$.

Answer: The most general harmonic polynomial of the desired form is $f(x, y) = ax^2 + bxy - ay^2$.

2.5.6. Solution 1: $f(x + iy) = u + iv$ is analytic in $D$. Then $uv$ is the imaginary part of the analytic function $\frac{1}{2}f^2(z)$; hence, $uv$ is harmonic.

Solution 2. Alternatively, we will check directly $uv$ satisfies the Laplace equation.

$$\frac{\partial^2 (uv)}{\partial x^2} = u_{xx}v + 2u_xv_x + uv_{xx}.$$  

Similarly

$$\frac{\partial^2 (uv)}{\partial y^2} = u_{yy}v + 2u_yv_y + uv_{yy}.$$  

Now

$$\frac{\partial (uv)}{\partial x} + \frac{\partial^2 (uv)}{\partial y^2} = u_{xx}v + 2u_xv_x + uv_{xx} + u_{yy}v + 2u_yv_y + uv_{yy} = (u_{xx} + u_{yy})v + u(v_{xx} + v_{yy}) + 2u_xv_x - 2v_xu_x = 0 + 0 + 0 = 0.$$  

because $u$ and $v$ are harmonic and the Cauchy-Riemann equations hold.

2.5.14. We check the Laplace equation. Since $\ln|f(z)| = \ln(u^2 + v^2)^{\frac{1}{2}} = \frac{1}{2} \ln(u^2 + v^2)$, we have

$$\frac{\partial}{\partial x} \ln|f(z)| = \frac{1}{u^2 + v^2} (uu_x + vv_x)$$  

and thus

$$\frac{\partial^2}{\partial x^2} \ln|f(z)| = \frac{\partial}{\partial x} \left( \frac{1}{u^2 + v^2} (uu_x + vv_x) \right) = \frac{(uu_{xx} + uu_x^2 + vv_{xx} + vv_x^2)}{u^2 + v^2} - 2 \frac{(uu_x + vv_x)^2}{(u^2 + v^2)^2}.$$  

Similarly,

$$\frac{\partial^2}{\partial y^2} \ln|f(z)| = \frac{(uu_{yy} + uu_y^2 + vv_{yy} + vv_y^2)}{u^2 + v^2} - 2 \frac{(uu_y + vv_y)^2}{(u^2 + v^2)^2} = \frac{(-uu_{xx} + uu_x^2 - vv_{xx} + vv_x^2)}{u^2 + v^2} - 2 \frac{(-uu_x + vv_x)^2}{(u^2 + v^2)^2}.$$
because \( u_x = v_y, u_y = -v_x, u_{xx} = -u_{yy}, v_{xx} = -v_{yy} \). Combining the last two formulas, we see that
\[
\frac{\partial^2}{\partial x^2} \ln |f(z)| + \frac{\partial^2}{\partial y^2} \ln |f(z)| = 0,
\]
i.e., \( \ln |f(z)| \) is harmonic.

3.1.4. Write \( p(z) = (z - z_1)(z - z_2) \cdots (z - z_n) \). Here the leading coefficient of \( p(z) \) is \( a_n = 1 \), and \( z_1, \ldots, z_n \) are the roots. Equating the constant terms on both sides, we see that
\[
a_0 = (-1)^n z_1 \cdots z_n.
\]
Thus \( 1 < |a_0| = |z_1| |z_2| \cdots |z_n| \), and so at least one root \( z_j \) must satisfy \( |z_j| > 1 \).

3.1.12. Suppose \( R_{m,n}(z) \) and \( r_{m,n}(z) \) agree at \( m+n+1 \) distinct points \( z_1, \ldots, z_{m+n+1} \), i.e.,
\[
R_{m,n}(z_j) = r_{m,n}(z_j)
\]
for each \( j = 1, 2, \ldots, m+n+1 \). Multiplying both sides by \( q(z)Q(z) \), we see that
\[
Pq(z_j) = PQ(z_j)
\]
for each \( j = 1, 2, \ldots, m+n+1 \). In other words, the polynomial \( f(z) = Pq(z) - PQ(z) \) vanishes at \( z_1, z_2, \ldots, z_{m+n+1} \). By our assumption \( Pq(z) \) and \( PQ(z) \) are polynomials of degree \( n+m \). Hence, \( f(z) \) is a polynomial of degree \( \leq m+n \). Since \( f(z) \) has at least \( m+n+1 \) distinct roots, it must be identically 0. That is, \( P(z)q(z) = P(z)Q(z) \) for all \( z \). Dividing both sides by \( q(z)Q(z) \), we obtain \( R_{m,n} = r_{m,n} \).

3.1.14. Write \( R(z) = r(z)/(z - z_0)^m \), where \( r(z) \) is a rational function such that \( r(z) \) is analytic at \( z_0 \) and \( r(z_0) \neq 0 \). Let us now compute the derivative of \( R(z) \):
\[
R(z)' = \frac{r(z)'}{(z - z_0)^m} - \frac{mr(z)}{(z - z_0)^{m+1}} = \frac{r(z)'(z - z_0) - mr(z)}{(z - z_0)^{m+1}}.
\]
The numerator, \( r(z)'(z - z_0) - mr(z) \) has neither a zero nor a pole at \( z = z_0 \). Thus \( R(z)' \) has a pole of order \( m+1 \) at \( z = z_0 \).

3.1.18. Following the hint, write
\[
R(z) = \frac{d_1(z - z_1)}{|z - z_1|^2} + \cdots + \frac{d_r(z - z_r)}{|z - z_r|^2}.
\]
Because we are assuming that each \( d_k \) is real and positive, \( \text{Im}(z) < 0 \), and \( \text{Im}(z_k) > 0 \), each vector \( \frac{d_1(z - z_k)}{|z - z_k|^2} \) points downward, so they cannot add to zero.

In other words, \( \text{Im}(z - z_k) < 0 \) and so
\[
\text{Im}(R(z)) = \text{Im}\left(\frac{d_1(z - z_1)}{|z - z_1|^2} + \cdots + \frac{d_r(z - z_r)}{|z - z_r|^2}\right) < 0.
\]
Thus \( R(z) \neq 0 \).
3.2.18. (a) Since \( \sin(0) = 0 \), we have
\[
\lim_{z \to 0} \frac{\sin(z)}{z} = \lim_{z \to 0} \frac{\sin z - \sin(0)}{z - 0} = \sin'(0) = \cos(0) = 1.
\]
(b) Similarly, since \( \cos(0) = 1 \),
\[
\lim_{z \to 0} \frac{\cos(z) - 1}{z} = \lim_{z \to 0} \frac{\cos z - \cos(0)}{z - 0} = \cos'(0) = -\sin(0) = 0.
\]

3.3.2. Formula 6:
\[
\log z_1 z_2 = \log |z_1 z_2| + i \arg(z_1 z_2) = \log |z_1| + \log |z_2| + i \arg(z_1 + z_2) = \\
\log |z_1| + i \arg(z_1) + \log |z_2| + i \arg(z_2) = \log(z_1) + \log(z_2).
\]

Formula 7:
\[
\log \frac{z_1}{z_2} = \log(z_1) + \log(1/z_2) = \log z_1 + \log |z_2^{-1}| + i \arg z_2^{-1} = \log z_1 - \log |z_2| - \\
i \arg z_2 = \log z_1 - \log z_2.
\]

3.3.6. For the complex logarithm \( \Log(z^2) = 2 \Log(z) \) does not hold.
For example, if we take \( z = -i \). Then \( \Log(z) = -\frac{\pi}{2} i \), \( z^2 = -1 \), and \( \Log(z^2) = \Log(-1) = \pi i \). Thus in this case
\[
\Log(z^2) = \pi i \neq 2(-\frac{\pi}{2} i) = 2 \Log(z).
\]