1.1.10. First we write the denominator in Cartesian form: \(6i - (1-2i) = (-1)+8i\). Now multiply top top and bottom by the conjugate, \(-1-8i\):

\[
\frac{2+i}{-1+8i} = \frac{(2+i)(-1-8i)}{(-1+8i)(-1-8i)} = \frac{6-17i}{65}.
\]

Now we can square this number:

\[
\left[\frac{2+i}{6i-(1-2i)}\right]^2 = \left(\frac{6-17i}{65}\right)^2 = \frac{36-17^2-2\cdot6\cdot17i}{65^2} = \frac{-253}{4225} - \frac{204}{4225}i.
\]

1.1.20

(a) \(iz = 4 - zi \Rightarrow 2iz = 4 \Rightarrow z = \frac{4}{2i} = -2i\).

(b) \(z = (1-5i)(1-z) = 1-5i - (1-5i) z \Rightarrow (1 + (1-5i)) z = 1-5i\). Dividing both sides by \(1 + (1-5i) = 2-5i\), we obtain,

\[
z = \frac{1-5i}{2-5i} = \frac{(1-5i)(2+5i)}{2^2+5^2} = \frac{27-5i}{29}.
\]

(c) One solution is \(z = 0\). If \(z \neq 0\), then we can divide both sides by \(z\). This yields \((2 - i) + 8z = 0\) and thus \(z = \frac{i-2}{8} = -\frac{1}{8} + \frac{1}{8}i\). Answer: The equation in part (c) has two solutions, \(z = 0\) and \(z = -\frac{1}{8} + \frac{1}{8}i\).

(d) \(z^2 + 16 = (z - 4i)(z + 4i)\). Thus there are two solutions, \(z = 4i\) and \(z = -4i\).

1.1.22. Factor \(z^4 - 16\):

\[z^4 - 16 = (z^2 - 4)(z^2 + 4) = (z - 2)(z + 2)(z - 2i)(z + 2i).\]

Thus there are four solutions: \(z = 2, -2, 2i\) and \(-2i\).

1.1.24. Let \(z = x + yi\), where \(y > 0\). Then

\[
\frac{1}{z} = \frac{1}{x + yi} = \frac{x - yi}{(x - yi)(x + yi)} = \frac{x -yi}{x^2 + y^2} = \frac{x}{x^2 + y^2} - \frac{y}{x^2 + y^2}i.
\]

We can now read off the imaginary part of \(\frac{1}{z}\); it is \(-\frac{y}{x^2 + y^2}\). Since \(y > 0\), this number is negative.

1.1.32. First compute

\[u = (a + b)(c + d) = ac + ad + bc + bd, \quad v = ac,\ \text{and} \quad w = bd.\]

This requires only three multiplications. Now the real part \(ac - bd\) and the imaginary part \(bc + ad\) of \((a + ib)(c + id)\) can be computed using only addition and subtraction: \(ac - bd = v - w\) and \(bc + ad = u - v - w\).
1.2.8. \( |z - 1|^2 = (z - 1)(\overline{z} - 1) = (\overline{z} - 1)(z - 1) = |\overline{z} - 1|^2 \Rightarrow |z - 1| = |\overline{z} - 1|. \)

1.2.16. Let \( z = x + iy \), \( 1 = |z|^2 = x^2 + y^2, z \neq 1. \)
\[
\begin{align*}
\frac{1}{1 - z} &= \frac{1}{1 - x - iy} = \frac{1 - x + iy}{(1 - x)^2 + y^2}, \\
\frac{1}{1 - x + iy} &= \frac{1 - x + iy}{(1 - x)^2 + y^2}.
\end{align*}
\]
Let us simplify the denominator: \( (1 - x)^2 + y^2 = 1 - 2x + x^2 + y^2 = 2 - 2x. \) Since \( z \neq 1 \) and \( |z| = 1 \), we see that \( |x| < 1 \) and thus \( 2 - 2x \) is a positive real number.
Thus the real part of \( \frac{1}{1 - z} \) is \( \frac{1 - x}{2 - 2x} = \frac{1}{2} \), as desired.

1.3.4. First consider the case where \( k \) is non-negative. Start with \( 1 = |z^0| = |z| \) (for \( k = 0 \)) and use the formula \( |z_1z_2| = |z_1||z_2| \) (which we proved in class) recursively (i.e., by induction on \( k \)): if we know that \( |z^{k-1}| = |z|^{k-1} \) then \( |z^k| = |z \cdot z^{k-1}| = |z||z^{k-1}| = |z||z|^{k-1} = |z|^k. \)

For negative \( k \) use the identity \( \frac{|z_1|}{|z_2|} = \frac{|z_1|}{|z_2|} \) proved in class, with \( z_1 = 1 \) and \( z_2 = z^{|k|}. \)

1.3.12. \( z = x + iy = r(\cos \phi + i \sin \phi), \) \( \cos \phi = \frac{x}{|z|}, \) \( \sin \phi = \frac{y}{|z|}, \) \( \phi = \arg(z). \)
(a) \( r = 6\sqrt{2}, \) \( \cos \phi = \sin \phi = -\frac{1}{\sqrt{2}} \Rightarrow \phi = \frac{3\pi}{4} \) and \( k \in \mathbb{Z} \)
(b) \( r = \pi, \) \( \cos \phi = -1, \) \( \sin \phi = 0 \Rightarrow \phi = \pi + 2\pi k, \) \( k \in \mathbb{Z} \)
(c) \( r = 10, \) \( \cos \phi = 0, \) \( \sin \phi = 1 \Rightarrow \phi = \frac{\pi}{2} + 2\pi k, \) \( k \in \mathbb{Z} \)
(d) \( r = 2, \) \( \cos \phi = \frac{\sqrt{3}}{2}, \) \( \sin \phi = -\frac{1}{2} \Rightarrow \phi = -\frac{\pi}{6} + 2\pi k, \) \( k \in \mathbb{Z}. \)

The principal values of the argument (taking values in the interval \( (-\pi, \pi] \)) are given as follows:
(a) \( \text{Arg}(-6 - 6i) = -\frac{3\pi}{4}, \)
(b) \( \text{Arg}(-\pi) = \pi, \)
(c) \( \text{Arg}(10i) = \frac{\pi}{2}, \)
(d) \( \text{Arg}(\sqrt{3} - i) = -\frac{\pi}{6}. \)

1.3.16. Since \( |z_1| - |z_2| \) is a real number, \( ||z_1| - |z_2|| = |z_1| - |z_2| \) or \( |z_2| - |z_1|. \)
Thus it suffices to show that
(i) \( |z_1| - |z_2| \leq |z_1 - z_2| \) and (ii) \( |z_2| - |z_1| \leq |z_1 - z_2|. \)
Note that (i) becomes (ii) if we interchange \( z_1 \) and \( z_2. \) Thus we only need to prove (i). To prove (i), start with the triangle inequality, \( |z + w| \leq |z| + |w| \) and substitute \( z_1 - z_2 \) for \( z \) and \( z_2 \) for \( w. \) We obtain
\[
|z_1| \leq |z_1 - z_2| + |z_2|.
\]
Subtracting \( |z_2| \) from both sides yields the inequality (i).