5.5.4. Start with
\[ \sin(w) = w - \frac{w^3}{3!} + \frac{w^5}{3!} - \ldots = \sum_{j=0}^{\infty} (-1)^j \frac{w^{2j+1}}{(2j+1)!}. \]
Substituting \(2z\) for \(w\) and dividing every term by \(z^3\), we obtain
\[ \frac{\sin(2z)}{z^3} = \sum_{j=0}^{\infty} (-1)^j \frac{2^{2j+1}}{(2j+1)!} z^{2j-2}. \]

5.5.6. Again, start with the power series
\[ \cos(w) = 1 - \frac{w^2}{2!} + \frac{w^4}{4!} - \ldots = \sum_{j=0}^{\infty} (-1)^j \frac{w^{2j}}{(2j)!}. \]
Now substitute \(\frac{1}{3z}\) for \(w\) and multiply each term by \(z^3\) to obtain the desired series:
\[ z^3 \cos\left(\frac{1}{3z}\right) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!} \frac{z^{2j}}{3^{2j}}. \]

5.6.2. Expanding \(\cos(z)\) into a Taylor series, we see that
\[ 2 \cos(z) - 2 + z^2 = -2 + z^2 + 2\left(1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \ldots\right) = 2 \frac{z^4}{4!} - 2 \frac{z^6}{6!} - \ldots \]
has a zero of order 4 at \(z = 0\). Thus \((2 \cos(z) - 2 + z^2)^2\) has a zero of order 8 at \(z = 0\). By a lemma proved in class, \(f(z) = 1/(2 \cos(z) - 2 + z^2)^2\) has a pole of order 8 at \(z = 0\).

5.6.17. (a) Suppose \(f(z) = a_1 z + a_2 z^2 + a_3 z^3 + \ldots\) is the Taylor series of \(f(z)\) at \(z_0 = 0\). Note that the constant term, \(a_0\) is 0, because \(f(0) = 0\). and \(a_1 = f'(0)\). Now for any \(z \neq 0\),
\[ F(z) = \frac{f(z)}{z} = a_1 + a_2 z + a_3 z^2 + \ldots. \]
Moreover, this equality also holds at \(z = 0\), since \(F(0) = f'(0) = a_1\). The right hand side is analytic in \(U\); hence, so is \(F(z)\).
(b) Applying the maximum modulus principle to $F(z)$ in the disc of radius $r < 1$ centered at the origin, we see that

$$|F(\zeta)| = \left| \frac{f(\zeta)}{\zeta} \right| \leq \max_{z \text{ on } C_r} \left| \frac{f(z)}{z} \right| = \max_{z \text{ on } \mathbb{C}} \left| \frac{f(z)}{r} \right| \leq \frac{1}{r}$$

(c) For $\zeta \neq 0$, $\left| \frac{f(\zeta)}{\zeta} \right| = |F(\zeta)| \leq \lim_{r \to 1-} \frac{1}{r} = 1$. Thus $f(\zeta) \leq |\zeta|$.

5.6.18. If $|f(z_0)| = |z_0|$ for $z_0 \neq 0$ then $|F(z_0)| = 1$. If $|f'(0)| = 1$ then $|F(0)| = 1$. In either case $|F|$ attains its maximal value maximum inside $U$. By the maximum modulus principle $F(z)$ must be constant in $U$. So $F(z) = F(z_0) = c$, where $|c| = 1$. Writing $c$ as $e^{i\theta}$ for some real number $\theta$, we obtain $f(z) = F(z)z = cz = e^{i\theta}z$.

6.1.4. If the Laurent series for $f(z)$ in a small punctured neighborhood of $z_0$ is

$$f(z) = \sum_{j=1}^{\infty} c_{-j}(z - z_0)^{-j} + \sum_{k=0}^{\infty} c_k(z - z_0)^k$$

then

$$f'(z) = \sum_{j=1}^{\infty} -jc_{-j}(z - z_0)^{-j-1} + \sum_{k=0}^{\infty} kc_k(z - z_0)^{k-1}.$$ 

The residue of $f'(z)$ at $z_0$ is the coefficient of $(z - z_0)^{-1}$ in this Laurent series. This coefficient is $0 \cdot c_0 = 0$.

6.3.1. Let us integrate $f(z) = \frac{1}{z^2 + 2z + 2}$ along the countour $\Gamma_\rho$ shown in Figure 6.4 on page 3.2.1 in the text. Note that

$$z^2 + 2z + 2 = (z + 1)^2 + 1 = (z + 1 + i)(z + 1 - i),$$

so $f(z)$ has only two singularities, $-1 + i$ and $-1 - i$. The countour $\Gamma_\rho$ consists of two curves a straight line segment along the $x$-axis, from $-\rho$ to $\rho$, and a semi-circle of radius $\rho$ in the upper half plane. Lemma 1 on page 322 tells us that

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2} = \int_{\Gamma_{\rho_0}} f(z)dz$$

for any $\rho > \sqrt{2}$. To evaluate the latter integral, we apply the residue theorem. The only singularity of $f(z)$ inside $\Gamma_\rho$ is $-1 + i$. The residue of $f(z)$ at this point is

$$\lim_{z \to -1+i} (z + 1 - i)f(z) = \lim_{z \to -1+i} \frac{1}{z + 1 + i} = \frac{1}{2i}.$$
Thus by the residue theorem,
\[ \text{p.v.} \int_{-\infty}^{\infty} \frac{dx}{x^2 + 2x + 2} = \int_{r_{\infty}} f(z)dz = 2\pi i \text{Res} \left( \frac{z}{z^2 + 2z + 2} ; -1 + i \right) = 2\pi i \frac{1}{2i} = \pi. \]

6.3.2. Arguing as above, we obtain
\[ \text{p.v.} \int_{-\infty}^{\infty} \frac{x^2 + 2}{x^2 + 2} dx = 2\pi i \text{Res} \left( \frac{x^2}{(x^2 + 2)^2} ; -1 + i \right) = \pi. \]

6.3.3. Once again, arguing as in Exercise 6.3.1, we obtain
\[ \text{p.v.} \int_{0}^{\infty} \frac{x^2 + 1}{x^4 + 1} dx = \frac{1}{2} \frac{2\pi i}{3} \left( \text{Res} \left( e^{\pi i/4} \right) + \text{Res} \left( e^{3\pi i/4} \right) \right) = \frac{\pi}{\sqrt{2}}. \]

6.3.11. Let the boundary of \( S_p \) be \( \gamma_1 + \gamma_2 + \gamma_3 \) where
\[
\begin{align*}
\gamma_1(t) &= t & 0 \leq t \leq \rho \\
\gamma_2(t) &= \rho e^{it} & 0 \leq t \leq 2\pi/3 \\
\gamma_3(t) &= -t e^{2\pi i/3} & -\rho \leq t \leq 0
\end{align*}
\]
then
\[ \int_{\gamma_1+\gamma_2+\gamma_3} \frac{dz}{z^3 + 1} = 2\pi i \text{Res} \left( e^{i\pi/3} \right) = \frac{2\pi i}{3} e^{-2\pi i/3} \]

\[ \lim_{\rho \to \infty} \int_{\gamma_2} \frac{dz}{z^3 + 1} = \lim_{\rho \to \infty} \int_{0}^{\rho} \frac{dx}{x^3 + 1} \]

\[ \lim_{\rho \to \infty} \int_{\gamma_2} \left| \frac{dz}{z^3 + 1} \right| \leq \lim_{\rho \to \infty} \frac{1}{\rho^3 - 1} \ell(\gamma_2) = \lim_{\rho \to \infty} \frac{\rho^2}{\rho^3 - 1} \frac{2\pi}{3} = 0 \]

On the other hand,
\[ \int_{\gamma_3} \frac{dz}{z^3 + 1} = \int_{0}^{\infty} \frac{e^{-2\pi i/3}}{-t^3 + 1} dt = -e^{2\pi i/3} \int_{0}^{\infty} \frac{dx}{x^3 + 1} \]

So
\[ \frac{2\pi i}{3} e^{-2\pi i/3} = \lim_{\rho \to \infty} \int_{\gamma_1+\gamma_2+\gamma_3} \frac{dz}{z^3 + 1} = (1 - e^{2\pi i/3}) \int_{0}^{\infty} \frac{dx}{x^3 + 1} \]

Thus
\[ \int_{0}^{\infty} \frac{dx}{x^3 + 1} = \frac{2\pi i}{3} e^{-2\pi i/3} \frac{1}{(1 - e^{2\pi i/3})} = \frac{2\pi \sqrt{3}}{9} \]