4.1.4. The parametrization \( z(t) \) is not admissible because \( z'(0) = 0 \). \( w(s) = s + is^2 \), \(-1 \leq s \leq 1\), is an admissible parametrization for the same curve.

4.1.8. Parametrization of \( \Gamma \):
\[
z(t) = \begin{cases} 
(1-2i)t - 1 & -1 \leq t \leq 0 \\
e^{i\pi(1-t)} & 0 \leq t \leq 1
\end{cases}
\]
Parametrization of \(-\Gamma\):
\[
z(t) = \begin{cases} 
1 + 2i & -1 \leq t \leq 0 \\
(-1+2i)t - 1 & 0 \leq t \leq 1
\end{cases}
\]

4.2.6. (a) \( \Gamma \) is parametrized by \( z(t) = 2e^{it}, 0 \leq t \leq 2\pi \). Thus
\[
\int_{\Gamma} zdz = \int_{0}^{2\pi} 2e^{it}2ie^{it}dt = 4i\int_{0}^{2\pi} e^{-it}e^{it}dt = 8\pi i.
\]
(b) \( \Gamma \) is parametrized by \( z(t) = 2e^{-it}, 0 \leq t \leq 2\pi \). Thus
\[
\int_{\Gamma} zdz = \int_{0}^{2\pi} 2e^{-it}(-i)e^{-it}dt = -8\pi i.
\]
(c) \( \Gamma \) is parametrized by \( z(t) = 2e^{-3it}, 0 \leq t \leq 2\pi \). Thus
\[
\int_{\Gamma} zdz = \int_{0}^{2\pi} 2e^{-3it}2(-3i)e^{-3it}dt = -24\pi i.
\]

4.2.10. \( C \) given by \( z(t) = \begin{cases} 
t & 0 \leq t \leq 1 \\
it + 1 - i & 1 \leq t \leq 2 \\
-t + 3 + i & 2 \leq t \leq 3 \\
-it + 4i & 3 \leq t \leq 4
\end{cases} \)
\[
\int_{C} z^2 dz = \int_{0}^{1} t^2 dt + \int_{1}^{2} (it + 1 - i)^2 dt + \int_{2}^{3} (-t + 3 + i)^2 (-1)dt + \int_{3}^{4} (-it + 4i)^2 (-i)dt = \\
\]
4.2.14. (a) $\ell(C) = 3 \cdot 2\pi$ and

$$\left|\frac{1}{z^2-i}\right| \leq \frac{1}{|z|^2-1} = \frac{1}{9-1}$$
on C. Thus

$$\left|\int_C \frac{dz}{z^2-i}\right| \leq \frac{3\pi}{4}.$$

(b) $\ell(C) = 2\pi$ and

$$\left|\frac{e^{3z}}{1+e^z}\right| = \frac{e^{3R}}{|1+e^z|} \leq \frac{e^{3R}}{e^R-1}$$
on C. Thus

$$\left|\int_C \frac{e^{3z}}{1+e^z}dz\right| \leq \frac{2\pi e^{3R}}{e^R-1}.$$

(c) $\ell(C) = \pi/2$ and

$$|\log z| = |0 + i| \leq \frac{\pi}{2}$$
on C. Thus

$$\left|\int_C \log z dz\right| \leq \frac{\pi^2}{4}.$$

(d) $\ell(C) = 1$ and

$$|e^{\sin z}| = e^{\Re e^{\sin z}} = e^0 = 1$$
on C. Thus

$$\left|\int_C e^{\sin z}dz\right| \leq 1.$$

4.3.4. False. For example, $f(z) = \frac{1}{z}$ is analytic at every point of the unit circle $\Gamma$ but

$$\int_{\Gamma} f(z)dz = 2\pi i.$$ 

Note that the Cauchy Integral Theorem assumes that $f(z)$ is analytic in some simply connected domain containing $\Gamma$.

4.3.12. Let $\Gamma$ be the line segment from $z_1$ to $z_2$. Then

$$|f(z_1) - f(z_2)| = \left|\int_{\Gamma} f'(z)dz\right| \leq M\ell(\Gamma) = M|z_2-z_1|$$

where $M = \max_{z \text{ on } \Gamma} |f'(z)|$.

4.4.10. The domains of analyticity are:

(a) $\mathbb{C}\backslash\{5i, -5i\}$

(b) $\mathbb{C}$

(c) $\mathbb{C}\backslash\{z: \Re(z) \leq -3, \Im(z) = 0\}$

In each case $f$ is analytic in a domain containing $|z| = 2$ and so the integral is 0 by Cauchy's integral theorem.

4.4.12. Otherwise the integral would be 0 by Cauchy's theorem.
4.5.4. (a) \( f(z) = \frac{z + i}{z + 2}, z_0 = 0, \int_C \frac{z + i}{z^3 + 2z^2}dz = 2\pi if'(0) = \pi/2 + \pi i. \)

(b) \( f(z) = \frac{z + i}{z + 2}, z_0 = -2, \int_C \frac{z + i}{z^3 + 2z^2}dz = 2\pi if(-2) = -\pi/2 - \pi i. \)

(c) \( \int_C \frac{z + i}{z^3 + 2z^2}dz = 0. \)

Reason: \( \frac{z + i}{z^3 + 2z^2} \) is analytic in a region \(|z-2i| < 2\) containing the interior of \(C\).

4.5.6. Let \( \Gamma_1 \) be the half circle from 3 to \(-3\) on the upper half plane and \( \Gamma_2 \) be the half circle from \(-3\) to 3 on the lower half plane such that \( \Gamma = \Gamma_1 + \Gamma_2 \). Let \( L \) be the line from \(-3\) to 3. Then

\[
\int_{\Gamma} \frac{e^{iz}}{(z^2 + 1)^2}dz = \int_{\Gamma_1 + L} \frac{e^{iz}}{(z^2 + 1)^2}dz + \int_{-L+\Gamma_2} \frac{e^{iz}}{(z^2 + 1)^2}dz = \\
\int_{\Gamma_1 + L} \frac{e^{iz}/(z+i)^2}{(z-i)^2}dz + \int_{-L+\Gamma_2} \frac{e^{iz}/(z+i)^2}{(z+i)^2}dz = 2\pi i(-e^{-1}i/2) + 2\pi i(0) = \pi/e.
\]

4.5.15. \( F(z) \) is continuous on \(|z| \leq 1\) and so \( G(z) \) is analytic on \(|z| < 1\).

\( G(0) = \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta^2}d\zeta = f'(0) = F(0) \). For \( z \neq 0 \)

\[
G(z) = \frac{1}{z} \left( \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta - z}d\zeta - \frac{1}{2\pi i} \int_{|\zeta|=1} \frac{f(\zeta)}{\zeta}d\zeta \right) \text{ (partial fractions)}
\]

\[
= \frac{1}{z} (f(z) - f(0)) = F(z)
\]

Therefore \( F(z) = G(z) \) on \(|z| < 1\) and so \( F \) is analytic on \(|z| \leq 1\).

4.6.4. From Theorem 20 it follows that

\[ |p^{(k)}(0)| = |k!a_k| \leq k!M/1 \]

Thus \( |a_k| \leq M \).

4.6.8. \( f(z)/3z^2 \) is analytic in \( 1 \leq |z| \leq 2 \) with \( |f(z)/3z^2| \leq 1 \) on \(|z| = 1 \) and on \(|z| = 2 \), hence by theorem 24, \( |f(z)/3z^2| \leq 1 \) in \( 1 \leq |z| \leq 2 \Rightarrow |f(z)| \leq 3|z|^2 \) in \( 1 \leq |z| \leq 2 \).

4.6.14. If \( f \) is non-zero then \( 1/f \) is analytic in \( D \) and \(|1/f(z)|\) attains its maximum value on the boundary of \( D \). In other words, \(|f(z)|\) attains its minimum value on the boundary of \( D \).

Example, showing that the assumption that \( f(z_0) \neq 0 \) for any \( z_0 \) in \( D \) is essential:

Suppose \( f(z) = z \) in \( D = \{|z| \leq 1\} \). Then \(|f|\) attains its minimum at \( z = 0 \) which is not on the boundary.