3.2.6. Proof of (8):

\[
\sin^2(z) + \cos^2(z) = \left( \frac{1}{2i} \left( e^{iz} - e^{-iz} \right) \right)^2 + \frac{1}{2} \left( e^{iz} + e^{-iz} \right)^2 = \\
-\frac{1}{4} \left( e^{2iz} - e^{-2iz} - 2 \right) + \frac{1}{4} \left( e^{2iz} + e^{-2iz} + 2 \right) = \\
\frac{1}{4} \left( -e^{2iz} - e^{-2iz} + 2 + e^{2iz} + e^{-2iz} + 2 \right) = \frac{1}{4} \cdot 4 = 1.
\]

Proof of (9):

\[
\sin(z_1) \cos(z_2) + \sin(z_2) \cos(z_1) = \\
\frac{1}{2i} \left( e^{iz_1} - e^{-iz_1} \right) \cdot \frac{1}{2} \left( e^{iz_2} + e^{-iz_2} \right) + \frac{1}{2i} \left( e^{iz_2} - e^{-iz_2} \right) \cdot \frac{1}{2} \left( e^{iz_1} + e^{-iz_1} \right) = \\
\frac{1}{4i} \left( e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{i(-z_1+z_2)} - e^{i(-z_1-z_2)} \right) + \\
\frac{1}{4i} \left( e^{i(z_1+z_2)} + e^{i(-z_2+z_1)} - e^{i(-z_1+z_2)} - e^{i(-z_1-z_2)} \right) = \\
\frac{1}{4i} \left( 2e^{i(z_1+z_2)} + e^{i(z_1-z_2)} - e^{i(-z_1+z_2)} - 2e^{i(-z_1-z_2)} \right) = \\
\frac{1}{2i} \left( e^{i(z_1+z_2)} - e^{i(-z_1-z_2)} \right) = \sin(z_1 + z_2).
\]

This shows that \( \sin(z_1 + z_2) = \sin(z_1) \cos(z_2) + \sin(z_2) \cos(z_1) \). Substituting \( -z_2 \) in place of \( z_2 \) into this formula, we obtain

\[
\sin(z_1 - z_2) = \sin(z_1) \cos(z_2) - \sin(z_2) \cos(z_1).
\]

3.2.12. (a) As we showed in the previous exercise, \( \cos^2(w) + \sin^2(w) = 1 \) for every complex number \( w \). Thus

\[
\cosh^2(z) - \sinh^2(z) = \cos^2(iz) - \left( \frac{1}{i} \sin(i z) \right)^2 = \cos^2(iz) + \sin^2(iz) = 1.
\]

(b) Similarly, using the identity

\[
\sin(w_1 + w_2) = \sin(w_1) \cos(w_2) + \sin(w_2) \cos(w_1)
\]

proved in the previous exercise, we obtain

\[
\sinh(z_1 + z_2) = \frac{1}{i} \sin(i z_1 + i z_2) = \frac{1}{i} \left( \sin(i z_1) \cos(i z_2) + \sin(i z_2) \cos(i z_1) \right) = \\
\sinh(z_1) \cosh(z_2) + \sinh(z_2) \cosh(z_1), \text{ as desired.}
\]
3.2.18. 
(a) Since \( \sin(0) = 0 \), we have
\[
\lim_{z \to 0} \frac{\sin(z)}{z} = \lim_{z \to 0} \frac{\sin z - \sin(0)}{z - 0} = \sin'(0) = \cos(0) = 1.
\]
(b) Similarly, since \( \cos(0) = 1 \),
\[
\lim_{z \to 0} \frac{\cos(z) - 1}{z} = \lim_{z \to 0} \frac{\cos z - \cos(0)}{z - 0} = \cos'(0) = -\sin(0) = 0.
\]
3.3.2. Formula 6:
\[
\log z_1 z_2 = \log |z_1 z_2| + i \arg(z_1 z_2) = \log |z_1| + \log |z_2| + i \arg(z_1 + z_2) = \\
\log |z_1| + i \arg(z_1) + \log |z_2| + i \arg(z_2) = \log(z_1) + \log(z_2).
\]
Formula 7:
\[
\log z_1 / z_2 = \log(z_1) + \log(1/z_2) = \log z_1 + \log |z_2|^{-1} + i \arg z_2^{-1} = \log z_1 - \\
\log |z_2| - i \arg z_2 = \log z_1 - \log z_2.
\]
3.3.6. For the complex logarithm \( \log(z^2) = 2 \log(z) \) does not hold.
For example, if we take \( z = -i \). Then \( \log(z) = -\frac{\pi}{2} i \), \( z^2 = -1 \), and \\
\( \log(z^2) = \log(-1) = \pi i \). Thus in this case
\[
\log(z^2) = \pi i \neq 2(-\frac{\pi}{2} i) = 2 \log(z).
\]
3.3.16. \( \Re(\log(z)) = \log(|z|) \), so the level curves are cencentric circles centered at the origin, shown in red below.
\( \Im(\log(z)) = \Arg(z) \), so the level curves are rays through the origin, shown in blue below.

At an intersection point of a ray with a circle, their slopes are orthogonal.
3.5.2. Setting $\alpha = 0$ in Definition 5, we obtain $z^0 = e^0 = 1$.

3.5.4. No. For example, $1^{1/n}$ assumes $n$ distinct values (the $n$th roots of unity). These are discussed in detail in Section 1.5.

3.5.9. Let $w = \cos^{-1}(z)$. Then $z = \cos(w) = \frac{e^{iw} + e^{-iw}}{2}$. Multiplying both sides by $2e^{iw}$, we obtain

$$e^{2iw} - 2ze^{iw} + 1 = 0.$$  

Using the quadratic formula, we can solve this equation for $e^{iw}$ as follows:

$$e^{iw} = z + \left(z^2 - 1\right)^{1/2}.$$  

Taking the log on both sides and dividing by $i$, we obtain

$$w = -i \log(z + (z^2 - 1)^{1/2}).$$

This proves formula (9).

To prove formula (11), differentiate both sides of $z = \cos(w)$ with respect to $z$. We obtain $1 = -\sin(w) \frac{dw}{dz}$ and thus

$$\frac{dw}{dz} = -\frac{1}{\sin(w)}.$$  

Since $\sin^2(w) + \cos^2(w) = 1$, and $\cos(w) = z$, we see that $\sin(w) = (1 - z)^{1/2}$, and the desired formula follows.