3.1.4. Write \( p(z) = (z-z_1)(z-z_2) \cdots (z-z_n) \), where \( z_1, \ldots, z_n \) are the roots. (Note that here the leading coefficient \( a_n \) is 1; see formula (8) on p. 101 in the book.)

Thus the constant term is \( a_0 = (-1)^n z_1 \cdots z_n \). Since \( 1 < |a_0| = |z_1||z_2| \cdots |z_n| \), at least one root \( z_j \) has to satisfy \( |z_j| > 1 \).

3.1.6. (a) Since conjugation commutes with sums, products and quotiens, we have
\[
z^n p(1/z) = z^n (\frac{a_n}{z^n} + \frac{a_{n-1}}{z^{n-1}} + \cdots + a_0) = p^*(z).
\]
(b) Suppose \( p(z_0) = 0 \) for some \( z_0 \neq 0 \). Then
\[
p^*(1/z_0) = (1/z_0)^n p(z_0) = 0.
\]
(c) Suppose \( |z| = 1 \), i.e., \( z = e^{i\theta} \) for some real number \( \theta \). Then \( z = 1/e^{-i\theta} = 1/z \) and \( |z|^n = 1 \). Thus
\[
|p^*(z)| = |z^n p(1/z)| = |p(z)| = |p(z)|.
\]

3.1.8. Write \( p(z) = (z-z_0)^m p_1(z) \) and \( q(z) = (z-z_0)^k q_1(z) \), where \( p_1(z) \) and \( q_1(z) \) do not vanish at \( z_0 \). Then \( p(z)q(z) = (z-z_0)^{m+k} p_1(z)q_1(z) \), where \( p_1(z)q_1(z) \) does not vanish at \( z_0 \). This shows that \( z_0 \) is a root of \( p(z)q(z) \) of order \( m+n \).

3.1.10. Write \( p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0 \). Then
\[
|p(z)| = |a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0| = |z|^n(|a_n + a_{n-1} \frac{1}{z} + \cdots + a_1 \frac{1}{z^{n-1}} + a_0 \frac{1}{z^n}|) = |z|^n(|a_n + g(z)|),
\]
where
\[
g(z) = a_{n-1} \frac{1}{z} + \cdots + a_1 \frac{1}{z^{n-1}} + a_0 \frac{1}{z^n}.
\]
Thus by the triangle inequality,
\[
|z^n(|a_n| - |g(z)|)| < |p(z)| < |z|^n(|a_n| + |g(z)|).
\]
It remains to show that \( |g(z)| < |a_n|/2 \) for \( |z| \) sufficiently large (i.e., whenever \( |z| > M \) for some real number \( M \)).

To prove this, we use the triangle inequality once again:
\[
|g(z)| \leq |a_{n-1}| \frac{1}{|z|} + \cdots + |a_1| \frac{1}{|z|^{n-1}} + |a_0| \frac{1}{|z|^{n}} = |a_{n-1}| \rho + \cdots + |a_1| \rho^{n-1} + |a_0| \rho^n,
\]
where \( \rho = \frac{1}{|z|} \). We know that
\[
\lim_{\rho \to 0} (|a_{n-1}| \rho + \cdots + |a_1| \rho^{n-1} + |a_0| \rho^n) = 0,
\]
i.e., \( |g(z)| \) becomes smaller than any fixed positive number for \( |z| \) sufficiently large. In particular, it becomes smaller than \( |a_n|/2 \), as desired.
Let $C$ be the maximal value of $|a_0|, |a_1|, \ldots, |a_{n-1}|$. Using the triangle inequality once again, we see that

$$|g(z)| \leq |a_{n-1}| \frac{1}{|z|} + \cdots + |a_1| \frac{1}{|z|^{n-1}} + |a_0| \frac{1}{|z|^n} \leq \frac{|C|}{|z|} (1 + \frac{1}{|z|} + \cdots + \frac{1}{|z|^{n-1}}).$$

If $|z| > 2$, then

$$1 + \frac{1}{|z|} + \cdots + \frac{1}{|z|^{n-1}} < 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} < 2$$

and thus

$$(1) \quad |g(z)| < 2 \frac{|C|}{|z|}.$$  

Now set $M = \max\{\frac{C}{4|a_n|}, 2\}$ and suppose $|z| > M$. Since $M \geq 2$, $(1)$ is valid, and since $M \geq \frac{4C}{|a_n|}$, we have

$$|g(z)| < 2 \frac{C}{|z|} < 2 \frac{|C|}{M} \leq 2 \frac{|C|}{\frac{4C}{|a_n|}} \leq \frac{|a_n|}{2},$$

as desired.

3.1.12. Suppose $R_{m,n}(z) = P(z)/Q(z)$ and $r_{m,n}(z) = p(z)/q(z)$ agree at $m+n+1$ distinct points $z_1, \ldots, z_{m+n+1}$. That is, $R_{m,n}(z_i) = r_{m,n}(z_i)$, for $i = 1, \ldots, m+n+1$. Clearing denominators, we see that $P(z_i)q(z_i) = p(z_i)Q(z_i)$ for each $i$. Thus $z_1, \ldots, z_{m+n+1}$ are roots of the polynomial $P(z)q(z) - p(z)Q(z)$. This polynomial is of degree at most $m+n$. Hence, this polynomial can only have $m+n+1$ roots if it is identically zero. We conclude that $P(z)q(z) - p(z)Q(z) = 0$ for every $z$. Equivalently, $P(z)/Q(z) = p(z)/q(z)$ for all $z$, as desired.

3.1.14. $R(z) = r(z)/(z - z_0)^m$, where $r(z_0) \neq 0$ and $r(z)$ has no pole at $z = z_0$. Then $R(z)' = \frac{r(z)'}{(z - z_0)^m} - \frac{m r(z)}{(z - z_0)^{m+1}} = \frac{r(z)'(z - z_0) - m r(z)}{(z - z_0)^{m+1}}$ and $r(z)'(z - z_0) - m r(z)$ has neither a zero a pole at $z = z_0$, so $R(z)'$ has a pole of order $m + 1$ at $z = z_0$.

3.1.18. Write

$$R(z) = \frac{d_1}{z - z_1} + \cdots + \frac{d_r}{z - z_r} = \frac{d_1 (z - z_1)}{|z - z_1|^2} + \cdots + \frac{d_r (z - z_r)}{|z - z_r|^2},$$

as the hint suggests. Note that $\frac{d_k}{|z - z_k|^2}$ is real and positive for each $k$. If $\text{Im}(z) < 0$ and $\text{Im}(z_k) > 0$, the imaginary part of $z - z_k$ is negative for each $k$. The same is true of $\frac{d_1 (z - z_1)}{|z - z_1|^2}$, because $\frac{d_1}{|z - z_1|^2}$ is real and positive for each $k$. We conclude that $R(z)$ cannot be 0 or such $z$. 
3.1.22. If we define the degree of the zero polynomial to be $-\infty$ and define
\[ -\infty + n = n + (-\infty) = -\infty \]
for every integer $n$, and
\[ -\infty + (-\infty) = -\infty , \]
then the identity
\[ \deg(p(z)q(z)) = \deg(p(z)) + \deg(q(z)) \]
will continue to hold even if $p(z) = 0$ or $q(z) = 0$ (or both).