In contrast to deterministic models, which predict certain regimes for a machine tool's stable equilibrium position, a stochastic model shows that noise can cause significant stochastic variation in the tool's position, leading to transitions from an equilibrium state to one of chatter. A reduced model—obtained by using a multiple-scales method adapted for stochastic dynamics—can capture the mechanism for these amplified oscillations via coherence resonance and provide an efficient computational method for the probability density of the machine tool's position.

This article gives the full model and shows how a multiple-scales approach leads to a reduced system that can describe the chatter's sustained amplification. A reduction provides an efficient semi-analytical approach for computing the dynamics: simulations of the behavior via the reduced system are a factor of \( \varepsilon^{-2} \) faster than simulations of the original system, where \( \varepsilon^{-1} \) is the oscillations' amplification factor. We'll also compare these results via computations of the probability densities and see how noise plays a role in the global dynamics.

Vibrations

Machine tool vibrations, commonly known as chatter, are essentially self-induced oscillations in a machining process, such as metal cutting, milling, or drilling. In addition to causing damage, chatter can also lower productivity and precision. Recent technological improvements, particularly those related to high-speed machining, have contributed to the varied sources of chatter. New directions in virtual machine tools rely on modeling and computational power, with chatter prediction serving as a crucial building block in the design of an efficient machining process.¹

Early efforts by J. Tlusty² and S.A. Tobias³ illustrate why these vibrations are inherent in the system: the force on the tool depends on the previous cut, as illustrated in Figure 1. Variations in the cut's thickness from the previous rotation can feed back into the system and excite further vibrations, thus the term regenerative chatter. Consequently, mathematical models for machine tool vibrations must involve a delay, representing the time of one revolution of the work piece. Delays naturally give rise to oscillatory and even chaotic behavior, depending on the model.⁴–¹⁰

The dynamics of machine tool vibrations have been studied from a variety of perspectives, including analysis, computation, and experiment.¹¹–¹⁹ Additional studies have considered variable speed,²⁰,²¹ nonlinear effects on frequency,¹⁹,²² and
the geometry of the cutting, which can change the bifurcation structure. Most analytical and computational studies focus on deterministic dynamics, but in contrast to deterministic models (which predict stability for the equilibrium position in certain parameter regimes), stochastic models capture a significantly different behavior, in which the interaction of noise and intrinsic oscillations can amplify otherwise-damped vibrations. It’s thus crucial to understand the variation in the tool’s position, which can be done via its probability density. In a stochastic setting, the collection of both computational and analytical methods for models with memory is limited. However, a multiple-scales method provides a reduced stochastic model that can capture the mechanism for the noise-induced amplification of the vibrations and provide an efficient computational approach for the probability density of the machine tool’s position.

A simple model demonstrates this phenomenon:

\[
dx = y \, dt
\]

\[
dy = (-2 \kappa y - x + c_1(x(t - \tau) - x(t)))dt + \delta dw(t), \quad (1)
\]

which is the linearized equation for variations \(x\) from a desired chip thickness due to vibrations. Note that this equation describes a spring with damping coefficient \(\kappa\), subject to both a force proportional to chip thickness \(x(t - \tau) - x(t)\) with coefficient \(c_1\), a non-dimensionalized material parameter, and noise, where \(w\) is a standard Brownian motion. Here, \(\tau\) is a delay due to the cutting tool’s rotation, as illustrated in Figure 1. Later, we’ll see that this linearized model captures the amplification of the intrinsic oscillations, which play a dominant role in transitions from equilibrium states to larger vibrations.

Figure 2 shows simulation results for Equation 1 in a setting in which the oscillations would decay to zero in the absence of noise—we choose the parameters in the region in which the equilibrium \((x = 0)\) solution is stable in the deterministic case. This phenomenon is known as \textit{autonomous stochastic resonance} or \textit{coherence resonance}: the presence of noise causes a resonance that sustains otherwise-damped oscillations. Although the sustained oscillations are noise-induced, they have a dominant regular frequency that can be verified by a standard computation of the power spectral density (not shown), which is strongly peaked at a particular frequency. The amplitude or envelope of these oscillations has significant variation—in fact, an order of magnitude larger than the actual noise level. The noise coefficient for both simulations is \(\delta = .05\), whereas the maximum amplitude regularly observed in the oscillations increases from approximately \(|x| = .2\) for \(c_1 = .07\) to \(|x| = .7\) for \(c_1 = .11\).

Researchers have studied coherence resonance in several contexts, with and without delays. However, few analytical methods have been developed for problems with delays and noise: computational methods are typically slow, and the computational error in the noise-sensitive regime can induce additional oscillations that interfere with the coherence resonance phenomenon. A
multiple-scales analysis can uncover the mechanism for noise-amplified oscillations by showing that the main factors are the noise’s resonance with the primary oscillation mode and its close proximity to the steady state’s stability boundary. In the context of regenerative chatter, the complex stability regions found for uniform cutting suggest that it can be advantageous to operate near these stability boundaries for steady, nonoscillatory behavior. However, analysis of the resonance effect also shows that vibrations can be amplified in parameter regimes near these stability boundaries, capturing the variability with an explicit noise amplification factor inversely proportional to the square root of the distance in parameter space from the stability boundary.24

The Model
Let’s begin with the one-degree-of-freedom model for machine tool dynamics, which is discussed in detail elsewhere:11

\[
\frac{d^2 z}{ds^2} + 2\alpha x \frac{dz}{ds} + \alpha^2 z = \frac{F(f)}{m}.
\]  

(2)

Here, \(z\) is the machine tool position, \(x\) is the damping factor, and \(\alpha = k/m\) is the ratio of stiffness \(k\) to \(m\) mass of the machine tool, so \(\alpha\) gives the natural frequency of the undamped system’s oscillations—that is, the machine tool’s natural vibrations. In general, the cutting force \(F\) is adjusted so that an equilibrium position \(z_0\) corresponds to a desired equilibrium chip thickness \(f_0\):

\[
\frac{d^2 z}{ds^2} + 2\alpha x \frac{dz}{ds} + \alpha^2 z = \frac{F(f)}{m}.
\]  

(3)

The functional form of the force \(F\) is determined experimentally, in which the cutting force is viewed as an empirical function of the physical parameters.2,3,11 Under ideal circumstances, the machine tool will stay uniformly at this equilibrium position \(z_0\). To consider variations about the equilibrium position \(z = z_0\), and how they translate into vibrations in the machine tool position, we must introduce the nondimensional variables

\[
z = z_0(1 + x), \quad t = \alpha x.
\]  

(4)

We write the force \(F\) in terms of actual chip thickness \(f = f_0 + (f - f_0)\), expressing the chip thickness variation \(f - f_0\) as the difference of the tool-edge position delayed by one rotation \(z(t - \tau)\) and the present position \(z(t)\). Substituting this into the expression for force (Equation 3), we can obtain a reasonable approximation by using a Taylor series of

\[F(f_0 + [z(t - \tau) - z(t)])\] about \(z(t - \tau) - z(t) = 0\), keeping terms up to \([z(t - \tau) - z(t)]^1\).11,23 To model variations in the material properties, we write \(K = K_0(1 + \eta)\), with \(\eta\) viewed as a percentage of \(K_0\). We model \(\eta\) with white noise that has a coefficient \(\delta < 1\) as a simple model for variations in the material properties encountered in the cutting process. The choice for \(\eta\) is convenient for analyzing delay-differential models,24 but we see a similar phenomena if we use colored noise. Substituting these expressions for the force and variation in Equation 2, and writing the second-order equation as a first-order system, we get

\[
dx = y dt
\]

\[
dy = \left(-2\kappa y - x + \sum_{i=1}^{3} c_i [x(t - \tau) - x(t)]^\gamma \right) dt
\]

\[
+ \delta dw + \sum_{i=1}^{3} c_i [x(t - \tau) - x(t)]^\gamma \delta dw,
\]  

(5)

where

\[
c_1 = \frac{3}{4} \frac{K_0 w}{(\kappa x f_0^{1/4})}, c_2 = -\frac{1}{8} \frac{c_1}{f_0},
\]

\[
c_3 = \frac{5}{96} \frac{c_1}{f_0^{1/3}}, \quad \tau = \frac{2\pi \alpha}{\Omega},
\]  

(6)

with \(\Omega\) representing the rotating work piece’s angular velocity. Complete details of the derivation appear elsewhere.23 We take \(\delta\) in the range .01 < \(\delta < .15\) as representative of typical variations in material properties. Equation 5 describes the variation \(x(1)\) as a percentage of the desired tool position \(z_0\). The key parameters are the delay \(\tau\), proportional to the inverse rotational frequency, \(c_1\), the non-dimensionalized material parameter, and \(\delta\), the percentage of variation in material properties. Next, we’ll examine a reduced model and compare it with numerical simulations for fixed damping \(\kappa = .05\), considering a range of \(c_1\) and \(\delta\) and a few different values for \(\tau\).25

Multiscale and Coherence Resonance
The multiscale analysis described in this section demonstrates how the stochastic amplitude of the nearly regular oscillations is amplified. One important ingredient in this phenomenon is the proximity of the parameters to the linear stability boundary of the equilibrium \(x = 0\). As Figure 2 shows, for the same noise level, the oscillations’ amplitude increases as \(c_1\) approaches this stability boundary (see Figure 3), even though the noise level remains the
same. Another important ingredient is the noise’s resonance with the tool position’s intrinsic oscillations. We can see this in the derivation of the amplitude equation, which also provides a reduced model for efficient computations of the dynamics.

**Linear Stability**

The stability boundary for the deterministic system has appeared in previous work. Here, we consider the system in Equation 5, linearized about the steady state equilibrium \( x = 0 \) (corresponding to no chatter, with \( \delta = 0 \)). This linearization yields

\[
\frac{d^2 x}{dt^2} + 2k \frac{dx}{dt} + x = c_1(x(t - \tau) - x(t)) .
\]

(7)

Substituting \( x = e^{\lambda t} \), we get the characteristic equation

\[
\lambda^2 + 2k\lambda + 1 = c_1(e^{\lambda \tau} - 1) .
\]

(8)

The stability boundary is determined by finding the curves satisfied by the equations for the real and imaginary parts of Equation 8 for \( \lambda = i\omega \). Figure 3 shows these curves in terms of the nondimensional parameters \( c_1 \) and \( \tau \). The linear stability criteria obtained from Equation 8 indicates that for a given value of \( \tau \), a critical value \( c_{1c} \) exists on the stability curve in Figure 3, as does a corresponding frequency \( \omega_c \) both obtained from Equation 8 with \( \lambda = i\omega_c \). This point corresponds to a Hopf bifurcation. For values of the material parameter \( c_1 < c_{1c} \), the equilibrium solution \( x = 0 \) is stable, and oscillations decay; for \( c_1 > c_{1c} \), the equilibrium is unstable, and oscillations with frequency \( \omega \) are sustained. Notice that in certain values of \( \tau \), the steady equilibrium state is stable over a larger range of \( c_1 \). We can write the dimensional parameters \( K \) and \( \Omega \) in terms of \( c_1 \) and \( \tau \), respectively; the stability boundary then shows how advantageous it is to operate at certain values of \( \Omega \), which should be stable for a larger range of the material parameter \( K \).

**Multiscale Analysis and Resonance**

The linear stability analysis given earlier illustrates that for values near the neutral stability curve—for example, for \( c_1 = c_{1c} + \varepsilon^2 c_{12} \) for \( \varepsilon << 1 \)—the eigenvalue in Equation 8 has the form \( \lambda = \varepsilon^2 r + i(\omega + \varepsilon^2 \omega_0) \). Here, we can obtain the real part \( \varepsilon^2 r \) and correction to the frequency \( \varepsilon^2 \omega_0 \) via a straightforward perturbation expansion of Equation 8 for \( c_1 = c_{1c} + \varepsilon^2 c_{12} \). The real part of \( \lambda \) is small \( (O(\varepsilon^2)) \), and \( r < 0 \) for \( c_{12} < 0 \)—that is, for values \( c_1 < c_{1c} \) below the stability boundary. Without noise \( (\delta = 0) \), we have a slow time scale \( T = \varepsilon^2 t \) for the decay of perturbations with critical frequency \( \omega_0 \). As we’ll see later, the introduction of a slow time scale is indeed critical for obtaining a reduced system that can describe the noise’s coherence resonance with oscillations.

An approximation of the solution of Equation 5 via a stochastic multiscale analysis captures the main features of the dynamics. Particularly in the region below the stability boundary for the steady state, we observe the following noise-induced oscillations by using the linearization of Equation 5,

\[
dx = \gamma dt
\]

\[
dy = (-2xy - x + c_1[x(t - \tau) - x(t)])dt + \delta dw
\]

\[
+ c_1[x(t - \tau) - x(t)] dw .
\]

(9)

Here, we’ve dropped the nonlinear terms—that is, we take \( c_2 = c_3 = 0 \), which is an appropriate approxi-
Figure 4. Comparison of the stationary probability density $p(x)$ for $\delta = .02$ computed from Equation 9. This linear system includes multiplicative noise (blue line) and uses the multiple-scales approximation, computed via Equation 11 (pink diamonds).

Figure 4 illustrates the comparison of the stationary probability density $p(x)$ for $\delta = .02$ as computed from Equation 9. The blue line represents the linear system that includes multiplicative noise, and the pink diamonds represent the multiple-scales approximation, both computed using Equation 11.

This linear system demonstrates the impact of the noise in the oscillation with a nearly deterministic frequency $\omega$, and the resulting reduced system gives an efficient means for computing an approximate probability density function over the long time scale $T$. The ansatz, or proposed, form for the equations for $A$ and $B$ is

$$\begin{align*}
\frac{dA}{dt} &= \begin{pmatrix} \psi_A \\ \psi_B \end{pmatrix} + \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix} \begin{pmatrix} \xi_1(T) \\ \xi_2(T) \end{pmatrix} \\
&+ \sum_{j=1}^{3} \sum_{i,j} \begin{pmatrix} A \\ B \end{pmatrix} \begin{pmatrix} d\beta_j(T) \\ d\beta_j(T) \end{pmatrix},
\end{align*}$$

where $\xi$ and $\beta$ are independent standard Brownian motions. The multiscale analysis’ goal is to derive the equations and Equation 11’s drift coefficients $\psi$, and $\psi$ and diffusion coefficients, the matrices $\Sigma$, and constants $\sigma$. The analysis also demonstrates that the ansatz for the form of the amplitude equations in Equation 11 is indeed consistent.

For clarity, let’s review the results here (details of the analysis appear in the “Appendix” sidebar).

$$\begin{align*}
(f_1^2 + f_2^2)\left(\omega^2 + 4\kappa^2\right)\psi_A &= \\
&- f_2\left[\left(c_1 A + c_1 B\right)\left(\omega \cos \omega t + 2\kappa \sin \omega t\right) + \left(2\kappa \omega \cos \omega t - \omega \sin \omega t\right)\left(c_1 A B + c_1 B + c_1 B\right) - \left(c_1 A B + c_1 A\right)\left(\omega \cos \omega t + 2\kappa \sin \omega t\right) - \left(c_1 A B + c_1 A\right)\left(-\omega B + 2\kappa A\right)\right],
\end{align*}$$

and

$$\begin{align*}
(f_1^2 + f_2^2)\left(\omega^2 + 4\kappa^2\right)\psi_B &= \\
&+ f_2\left[\left(c_1 A + c_1 B\right)\left(\omega \cos \omega t + 2\kappa \sin \omega t\right) + \left(2\kappa \omega \cos \omega t - \omega \sin \omega t\right)\left(c_1 A B + c_1 B + c_1 B\right) - \left(c_1 A B + c_1 A\right)\left(\omega \cos \omega t + 2\kappa \sin \omega t\right) - \left(c_1 A B + c_1 A\right)\left(-\omega B + 2\kappa A\right)\right].
\end{align*}$$
APPENDIX

Let’s combine two sets of equations for $dx$ and $dy$, which yields equations for the drift coefficients $\psi_d$ and $\psi_B$ and diffusion coefficients $\sigma_d$ in the main text’s Equation 11. The first uses Ito’s formula, which relates $dx$ and $dy$ to $dA$ and $dB$,

$$dx = \frac{\partial x}{\partial t} dt + \frac{\partial x}{\partial A} dA + \frac{\partial x}{\partial B} dB,$$

$$dy = -\omega \sin \omega t dt + \cos \omega t \{\psi_d d\omega T + \sigma_d d\xi_1(T) + \sigma_{d2} d\xi_2(T)\} + \sin \omega t \{\psi_B d\omega T + \sigma_B d\xi_1(T) + \sigma_{B2} d\xi_2(T)\},$$

(A)

and similarly for $dy$. Terms such as $d^2x/dA^2$ and $d^2x/dB^2$ don’t appear here, due to Equation 10. Substitution of Equation 10 into Equation 9 gives a second expression involving $dx$ and $dy$:

$$dx = -\omega \sin \omega t dt$$

$$dy = \frac{2\omega \sin \omega t - \cos \omega t}{\omega^2 + 4\kappa^2} dt + \frac{A(T-e^2\tau) - A(T)}{\epsilon^2} dt + \frac{B(T-e^2\tau) - B(T)}{\epsilon^2} dt + \sin \omega t \{\psi_d d\omega T + \sigma_d d\xi_1(T) + \sigma_{d2} d\xi_2(T)\} + \sin \omega t \{\psi_B d\omega T + \sigma_B d\xi_1(T) + \sigma_{B2} d\xi_2(T)\},$$

(B)

Only the additive noise is retained in these equations. As Figure 4 verifies, we have good agreement between the system-approximated diffusion terms with additive noise only and the density for the system with both additive and multiplicative noise. Thus we’ve confirmed that additive noise plays the dominant role in the dynamics for the regime considered in the main text, so we can neglect the multiplicative noise terms in the analysis.

To determine the equation for $A$ and $B$, we equate $dx$ and $dy$ from Equations A and B, write $c_1 = c_{1c} + \epsilon^2 c_{12}$ for $c_{12} < 0$ in Equations A and B, and use a perturbation expansion for $\epsilon < \ll 1$ for the drift terms. The leading order ($O(1)$) terms cancel because we have the linear homogeneous equation with $c_1 = c_{1c}$ and Equation 10, meaning we treat $A$ and $B$ as constants to the leading order in the multiscale expansion. The next order corrections are $O(\epsilon^2)$, leaving the drift terms with coefficient $\epsilon^2 dt = d\tau$. The leading order diffusion (noise) terms have coefficient $\delta << 1$.

The resulting system includes terms that vary on both the $t$ and $T$ time scales. To obtain the coefficients in Equation 10, which are functions of $T$ only, we use a projection on the fast oscillations with frequency $\omega$, defined as the inner product of $(dx, dy)$ with the solutions to the adjoint of the homogeneous linear equation with $c_1$ at the critical value of $c_{1c}$ for deterministic systems. $1$ is typically used in multiple-scales analysis for deterministic systems, the inner product has the form

$$\int_0^{2\pi/\omega} dx(Eq. A) \psi_j + dy(Eq. A) \psi_j dt = \int_0^{2\pi/\omega} dx(Eq. B) \psi_j + dy(Eq. B) \psi_j dt$$

$j = 1, 2,$

(C)

where subscripts refer to Equations A and B, and $(\psi_j, \psi_j)$ for $j = 1, 2$, are the two solutions to the adjoint linear problem for Equation 9 with $\delta = 0$,

$$\sigma_{11} = -\sigma_{12}, \sigma_{22} = \sigma_{11},$$

$$\sigma_{21} = \sigma_{12}$$

(14)

For the purposes of our discussion here, we show only the additive noise terms. Note that these terms have a coefficient of $\delta / \epsilon$, indicating the $\epsilon^{-1}$ amplification of the noise is due to parameters in close proximity to the equilibrium position’s stability boundary. The drift coefficients $\psi_d$ and $\psi_B$ include terms that depend on the delayed time $T = e^2 \tau$, so, in general, we must determine the behavior of $A$ and $B$ numerically from Equation 11. As described in the Appendix, the terms in Equation 11 with coefficient $\Sigma$ are higher-order corrections, so we don’t include them here.

Figure 4 compares the numerical calculation for the stationary probability density $p(y)$ obtained from Equation 9 (with both multiplicative and additive noise) and the multiscale approxi-
The good agreement shows that in this regime, the additive noise plays the dominant role in amplifying the oscillations of $x$, which are a percentage of the equilibrium $z_0$. The variance is an order of magnitude larger than the noise factor $\gamma$ (in this case, $\gamma = 0.02$), reflecting amplification of the noise for parameter values near the stability boundary in Figure 3, in which the equilibrium steady state loses stability to the chatter oscillation. Although the parameter values are in the stability regime for the deterministic equilibrium, the oscillations are sustained and have an amplification factor related to $\gamma$, where $\varepsilon$ measures proximity to the stability boundary.

We obtain the approximate stationary densities by running simulations over a sufficiently long time for 5,000 realizations. Note that the calculations using the multiscale approximation are for $A$ and $B$ on the $T = \varepsilon^2 t$ scale, so the computations for the multiscale approximation are a factor of $O(\varepsilon^2)$ faster than the full system simulation on the $t$ scale. The numerical simulations used both the Euler-Maruyama equation and the multistep methods for stochastic delay differential equations. By using a higher-order method, we can verify that the oscillations aren’t a result of the numerical method.

Additional simulations appear elsewhere for ranges of $\delta$ and $c_1$; the behavior of the densities for these other cases is similar.

**Implications for Oscillations in the Nonlinear System**

Figure 5 illustrates the implications of coherence resonance in the linear system for realizations of the fully nonlinear system. A typical bifurcation structure for the full system is that of a subcritical bifurcation, which is found in several studies of deterministic models for machine tool vibrations. For this setting in the absence of noise, small perturbations from the zero state are damped because small oscillations are unstable. In the stochastic setting for parameter values near the bifurcation, the oscillations are sustained and have an amplification factor related to $\varepsilon$, where $\varepsilon$ measures proximity to the stability boundary. Once these oscillations are large enough, the nonlinear system can make the transition to a branch of large-amplitude behavior, as shown in Figure 5a. The coherence resonance in the stochastic system thus provides a mechanism by which the tool behavior can shift to a nonlinear mode, and for large enough $x$, the tool can even lose contact with the work.
Indeed, experiments show sustained oscillations for the transition to oscillations with (green line), so that eventually the system makes to zero (red line), pushing the system to steady

sence of noise, the system’s oscillations damp out zero solution, with and without noise. In the ab-

system for parameters in the stability range for the linear system and compares the realization of the linear mode with large oscillations for the full nonlinear system and compares the realization of the linear mode with large oscillations for the full nonlinear system.

We leave a complete exploration of the fully nonlinear systems with noise and memory. The irregularity; the additive noise doesn’t influence the dynamics as much because it’s relatively small. We leave a complete exploration of the fully nonlinear noisy system for future work; it depends on the development of new approaches for analyzing fully nonlinear systems with noise and memory.

We’ve seen that certain critical combinations of noise levels and parameter regimes near stability boundaries can allow and even promote transitions between different bistable states. Even if the system is operating in a regime in which we might expect the equilibrium state to be linearly stable, small noisy perturbations can be amplified through coherence resonance. This amplification is of central importance in practical set-

piece intermittently. In this case, we must analyze a more complex model that allows for periods during which the tool might skip.14

Figure 5b demonstrates the transition to a nonlinear mode with large oscillations for the full nonlinear system and compares the realization of the system for parameters in the stability range for the zero solution, with and without noise. In the absence of noise, the system’s oscillations damp out to zero (red line), pushing the system to steady state. However, even with small noise (in this case, \( \delta = .05 \)), we see that small oscillations are amplified (green line), so that eventually the system makes the transition to oscillations with \( O(1) \) amplitude. Indeed, experiments show sustained oscillations for parameters in the deterministic stable regime near the stability boundary,14 suggesting that noise could contribute to the dynamics.

Once the system makes a transition to large oscil-
culations, the multiplicative noise contributes to the irregularity; the additive noise doesn’t influence the dynamics as much because it’s relatively small. We leave a complete exploration of the fully nonlinear noisy system for future work; it depends on the development of new approaches for analyzing fully nonlinear systems with noise and memory.

**Reference**

tings, where it can cause transitions to large-amplitude states or chatter.

A multiple-scales analysis yields a means for efficient computation of the probability density for a machine tool’s position. This density is the basis for quantities that can describe the system’s state: density, for example, can help us calculate the likelihood that the variations exceed a threshold, corresponding to the transition to chatter, or estimate the time in which such a transition might occur. In this case, the amplification factor indicates that this transition can occur on an \( O(1) \) time scale, so we can use the analysis to predict parameter regimes in which such a transition is likely to be observed. Generalizations of the approach show promise in biological modeling—in particular, studies of periodic recurrence of epidemics due to random variations in populations and in noise-induced neuronal activity and synchronization. It has also been applied in nonlinear settings, such as logistic models and large-amplitude oscillations in neuronal bursting dynamics.

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**References**


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