Exercise: Determine whether the following converge absolutely, converge conditionally or diverge.

(a) \( \sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n - 1} \).

Solution: This series diverges. To see why, note that since \( \lim_{n \to \infty} \frac{3n}{4n - 1} = \frac{3}{4} \), \( \lim_{n \to \infty} \frac{(-1)^n 3n}{4n - 1} \) does not exist. So, since the terms do not go to zero, it follows from the Divergence Test that the series diverges.

(b) \( \sum_{n=10}^{\infty} \frac{\cos(n\pi)}{\log(n)} \).

Solution: This series converges conditionally. First, since \( \cos(n\pi) = (-1)^n \), the series is an alternating series. So, since
\[
\left| \frac{\cos(n\pi)}{\log(n)} \right| = \frac{1}{\log(n)} > \frac{1}{n}
\]
for all \( n \geq 10 \), and since \( \sum_{n=10}^{\infty} \frac{1}{n} \) diverges (harmonic series), \( \sum_{n=10}^{\infty} \frac{\cos(n\pi)}{\log(n)} \) must also diverge by the Comparison Test.

Now, \( \frac{1}{\log(n)} > 0 \) for all \( n \geq 10 \) and \( \lim_{n \to \infty} \frac{1}{\log(n)} = 0 \). Moreover, since \( \log(n) \) is increasing in \( n \), \( \frac{1}{\log(n)} \) is decreasing in \( n \), i.e. \( \frac{1}{\log(n)} > \frac{1}{\log(n+1)} \) for all \( n \geq 10 \). So, \( \sum_{n=10}^{\infty} \frac{\cos(n\pi)}{\log(n)} \) converges by the Alternating Series Test, and therefore converges conditionally.

(c) \( \sum_{n=4}^{\infty} \frac{(-1)^n n}{2^n} \).

Solution: This series converges absolutely. To see this, let \( a_n = \frac{n}{2^n} \). Then,
\[
\frac{a_{n+1}}{a_n} = \frac{\frac{n+1}{2^{n+1}}}{\frac{n}{2^n}} = \frac{n+1}{2n}.
\]
Therefore,
\[
\lim_{n \to \infty} \frac{a_{n+1}}{a_n} = \lim_{n \to \infty} \frac{n+1}{2n} = \frac{1}{2} < 1.
\]
So, it follows from the Ratio Test that the series \( \sum_{n=4}^{\infty} \frac{(-1)^n n}{2^n} \) converges absolutely.
(d) \[ \sum_{n=7}^{\infty} \frac{8}{(2n)!} - 1. \]

Hint: This series converges absolutely. Use the Limit Comparison Test.

(e) \[ \sum_{n=2}^{\infty} \frac{4n + 1}{\sqrt{n^3 - 7}}. \]

Hint: This series diverges. Use the Comparison Test.

(f) \[ \sum_{n=3}^{\infty} n^{1/n}. \]

Hint: This series diverges. Show that \[ \lim_{n \to \infty} n^{1/n} = 1 \] (second hint: to do this, it may be helpful to first compute the limit \[ \lim_{n \to \infty} \log(n^{1/n}) \]) and use the Divergence Test.