1. By repeated applications of the Product Rule, find the first five derivatives of \( f(x)g(x) \). Do you notice anything about the pattern? Without calculating it, can you make a guess as to what is the sixth derivative of \( f(x)g(x) \)?

**Hint:** See written assignment #4, question 3(a).

2. (a) Suppose \( f, g \) and \( h \) are all differentiable at \( x \). Find the derivative of \( f(x)g(x)h(x) \). Can you explain why this is plausible using a geometric argument like the one used to justify the Product Rule?

**Solution:** To differentiate \( f(x)g(x)h(x) \), we let \( F(x) = f(x)g(x) \). Then

\[
\frac{d}{dx} \left( f(x)g(x)h(x) \right) = \frac{d}{dx} \left( F(x)h(x) \right) = F'(x)h(x) + F(x)h'(x) \quad \text{(by the Product Rule)}.
\]

But, since \( F(x) = f(x)g(x) \), we can use the Product Rule again to get that \( F'(x) = f'(x)g(x) + f(x)g'(x) \). So,

\[
\frac{d}{dx} \left( f(x)g(x)h(x) \right) = \left( f'(x)g(x) + f(x)g'(x) \right) h(x) + \left( f(x)g(x) \right) h'(x) = f'(x)g(x)h(x) + f(x)g'(x)h(x) + f(x)g(x)h'(x).
\]

You can use a geometric argument (like the one we used in class to justify the Product Rule) to explain why this is plausible, but it more complicated. Instead of looking at the difference in area between two rectangles, in this case we would look at the difference in volume between two rectangular prisms. This difference in volume breaks up into 7 pieces:

\[
(f(x + h) - f(x))g(x)h(x), \quad (f(x + h) - f(x))(g(x + h) - g(x))h(x), \quad f(x)(g(x + h) - g(x))h(x), \quad (f(x + h) - f(x))g(x)(h(x + h) - h(x)), \quad f(x)(g(x + h) - g(x))(h(x + h) - h(x)), \quad f(x + h) - f(x)(g(x + h) - g(x))(h(x + h) - h(x)).
\]

Each of the terms in the second row is such that if you divide it by \( h \) and take the limit as \( h \) goes to zero, then you will get zero. When you do the same to the terms on the right, their limit will be one of the terms in the formula derived above. For example,

\[
\lim_{h \to 0} \frac{(f(x + h) - f(x))g(x)h(x)}{h} = f'(x)g(x)h(x).
\]

(b) Make a conjecture as to what is the derivative of \( f_1(x)f_2(x)\cdots f_n(x) \), where \( f_1, f_2, \ldots, f_n \) are differentiable and \( n \) is a positive integer. What is required to prove this conjecture?
Based on the question above, a suitable conjecture might be

\[
\frac{d}{dx} \left( f_1(x) \cdots f_n(x) \right) = f'_1(x) f_2(x) \cdots f_n(x) + f_1(x) f'_2(x) f_3(x) \cdots f_n(x) + \cdots + f_1(x) \cdots f_{n-1}(x) f'_n(x),
\]
and indeed this is the correct formula. We will prove this formula using mathematical induction. First, we know that the formula is true when \( n = 2 \) because this is just the Product Rule:

\[
\frac{d}{dx} \left( f_1(x) f_2(x) \right) = f'_1(x) f_2(x) + f_1(x) f'_2(x),
\]
which we proved in class. Next, we assume that the formula is true when \( n = k \) for some integer \( k \), and we show that this implies that it must also be true when \( n = k + 1 \). The formula when \( n = k \) is

\[
\frac{d}{dx} \left( f_1(x) \cdots f_k(x) \right) = f'_1(x) f_2(x) \cdots f_k(x) + \cdots + f_1(x) \cdots f_{k-1}(x) f'_k(x).
\]

To take the derivative of \( f_1(x) \cdots f_k(x) \), we will think of \( f_1(x) \cdots f_k(x) \) as one function and use the Product Rule:

\[
\frac{d}{dx} \left( f_1(x) \cdots f_{k+1}(x) \right) = \frac{d}{dx} \left( (f_1(x) \cdots f_k(x)) f_{k+1}(x) \right) = (f_1(x) \cdots f_k(x))' f_{k+1}(x) + (f_1(x) \cdots f_k(x)) f'_{k+1}(x).
\]

By assumption, \( (f_1(x) \cdots f_k(x))' \) satisfies the formula, so we may substitute it into the above equation to get

\[
\frac{d}{dx} \left( f_1(x) \cdots f_{k+1}(x) \right) = \left( f'_1(x) f_2(x) \cdots f_k(x) + \cdots + f_1(x) \cdots f_{k-1}(x) f'_k(x) \right) f_{k+1}(x) + (f_1(x) \cdots f_k(x)) f'_{k+1}(x)
\]

\[= f'_1(x) f_2(x) \cdots f_k(x) f_{k+1}(x) + \cdots + f_1(x) \cdots f'_k(x) f_{k+1}(x) + f_1(x) \cdots f_k(x) f'_{k+1}(x),\]

which is just our formula when \( n = k + 1 \). So, we know the formula is true when \( n = 2 \), and that if it is true for \( n = k \), then it is also true for \( n = k + 1 \). This allows us to conclude that the formula must also be true for \( n = 3 \) and \( n = 4 \), and, in fact, for any positive integer \( n \).

3. Suppose \( f(1) = 3 \) and \( f'(1) = -5 \). Show that the line tangent to \( x^2 f(x) \) at \( x = 1 \) always lies above the line \( y = x \). (Note: There was a typo on the handout! Originally, the problem said “below” instead of “above”. Sorry!)

**Solution:** Our first step is to find the equation of the tangent line. By the Product Rule,

\[
\frac{d}{dx} \left( x^2 f(x) \right) = 2xf(x) + x^2 f'(x),
\]
so we see that the line tangent to \( x^2 f(x) \) at \( x = 1 \) has slope \( 2 \cdot f(1) + 1 \cdot f'(1) = 6 - 5 = 1 \). Also, \( x^2 f(x) \) passes through the point \((1, 3)\). So, the equation of the tangent line is given by

\[
y - 3 = 1(x - 1) \quad \text{or} \quad y = 2 + x.
\]

Now, we see that the tangent line and the line \( y = x \) both have slope 1, and so they are therefore parallel lines. Also, the tangent line has \( y \)-intercept 2 whereas the line \( y = x \) has \( y \)-intercept 0, so it follows that the tangent line always lies above the line \( y = x \).