Note: For all of the questions below, derivatives should be calculated using the *limit definition* rather than any differentiation rules you may already know.

1. Let

\[ f(x) = \begin{cases} 
3x^2 + 5, & \text{if } x \geq 6 \\
2x, & \text{if } x = 5 \\
0, & \text{if } x < 3
\end{cases} \]

For which values of \( x \) does \( f'(x) \) exist?

**Solution:** If \( x > 6 \), then, since we’re considering the limit as \( h \) goes to zero, we can assume \( h \) is close enough to zero so that \( x + h > 6 \). So

\[
\lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{3(x + h)^2 + 5 - (3x^2 + 5)}{h} = \lim_{h \to 0} \frac{3x^2 + 6xh + h^2 - 3x^2}{h} = 6x.
\]

Therefore, \( f'(x) \) exists for \( x > 6 \).

Now, notice that if \(-1 < h < 0\), \( f(6 + h) \) is undefined. So, \( f'(6) = \lim_{h \to 0} \frac{f(6+h) - f(6)}{h} \) cannot exist.

Similarly, for \(-2 \leq h < 1, h \neq 0\), \( f(5 + h) \) is undefined, so \( f'(5) \) cannot exist either.

Finally, for \( x < 3 \), taking \( h \) close enough to zero so that \( x + h < 3 \) we have that

\[
\lim_{h \to 0} \frac{f(3 + h) - f(3)}{h} = \lim_{h \to 0} \frac{0}{h} = 0.
\]

Therefore, \( f'(x) \) exists for \( x \in (-\infty, 3) \cup (6, \infty) \).

2. The graph of a function \( f \) is shown below.
(i) List all values of $x$ at which $f$ is not continuous.
(ii) List all values of $x$ at which $f$ is not differentiable.

Solution:
(a) The points (in $(0, 7)$) of discontinuity are: $x = 3, 4, 5$.
(b) The points (in $(0, 7)$) of non-differentiability are $x = 2, 3, 4, 5, 6$.

3. Sketch the graph of a function on $[0, 4]$ satisfying all of the following criteria:

(i) $f$ is not differentiable at $x = \frac{1}{2}, \frac{3}{2}$.
(ii) $f$ is continuous everywhere except $x = 0, 1, 2, 3$.
(iii) $\lim_{x \to 0} f(x) = 0$.
(iv) $0 \leq f(x) \leq 2$ for all $x \in [0, 4]$.

Solution: Here is one example of the graph of such a function:

![Graph of a function](image)

4. Can you think of a function that is differentiable everywhere, but whose derivative is not differentiable everywhere?

**Hint:** Consider the function defined in question #3 in the practice problems from September 17th.

5. Can you think of a function that is differentiable everywhere, but whose derivative is not continuous everywhere?

This is difficult problem! I won’t post a solution to this now because we will return to it in a few weeks. For now, it is really just something to think about, and it is enough for you to know that such functions exist.