Exercise Consider the function $f(x) = \frac{1}{x^2}$.

(a) Show that $\lim_{x \to 0} \frac{1}{x^2} \neq 1$.

(b) Show that $\lim_{x \to 0} \frac{1}{x^2} \neq C$. What can you conclude from this?

(c) Show that $\lim_{x \to 0} \frac{1}{x^2} = \infty$.

Solutions:

(a) We’ll show that $\lim_{x \to 0} \frac{1}{x^2} \neq 1$ by showing that we can’t even make $\frac{1}{x^2}$ always be within $\varepsilon = 1$ of 1 by making $x$ sufficiently close to 0. That is, we want to show that, no matter how small $\delta$ is, there is always a number $x \in (-\delta, \delta)$, $x \neq 0$, for which $\frac{1}{x^2} > 1 + 1$.

Rough work:

$$\frac{1}{x^2} > 2 \iff x^2 < \frac{1}{2} \iff -\sqrt{2} < x < \sqrt{2}.$$  

For any $\delta > 0$, let $x$ be the smaller of $\frac{1}{\sqrt{2}}$ and $\frac{\delta}{2}$. Then $x \in (-\delta, \delta)$, but $\frac{1}{x^2} > 2$ (since $-\sqrt{2} < x < \sqrt{2}$ as well). Hence, $\frac{1}{x^2}$ does not become arbitrarily close to 1 as $x$ approaches 0 since it is not even within $\varepsilon = 1$ of 1.

(b) Again, we’ll show that $\lim_{x \to 0} \frac{1}{x^2} \neq C$ by showing that we can’t even make $\frac{1}{x^2}$ always be within $\varepsilon = 1$ of $C$ by making $x$ sufficiently close to 0.

Rough work:

$$\frac{1}{x^2} > C + 1 \iff x^2 < \frac{1}{C + 1} \iff -\frac{1}{\sqrt{C + 1}} < x < \frac{1}{\sqrt{C + 1}}.$$  

For any $\delta > 0$, let $x$ be the smaller of $\frac{1}{\sqrt{C + 1}}$ and $\frac{\delta}{2}$. Then $x \in (-\delta, \delta)$, but $\frac{1}{x^2} > C + 1$ (since $-\frac{1}{\sqrt{C + 1}} < x < \frac{1}{\sqrt{C + 1}}$ as well). Hence, $\frac{1}{x^2}$ does not become arbitrarily close to $C$ as $x$ approaches 0 since it is not even within $\varepsilon = 1$ of $C$.

This implies that $\lim_{x \to 0} \frac{1}{x^2}$ DNE since, if the limit did exist, it would have to be equal to some number $C$, which we just showed isn’t the case.

(c) To show that $\lim_{x \to 0} \frac{1}{x^2} = \infty$ we need to show that $\frac{1}{x^2}$ gets arbitrarily large, provided $x$ is sufficiently close to 0. That is, we need to show that for any number $N > 0$, there is a $\delta > 0$ so that $\frac{1}{x^2} > N$ provided $x \in (-\delta, \delta)$, $x \neq 0$.

Rough work:

$$\frac{1}{x^2} > N \iff x^2 < \frac{1}{N} \iff -\frac{1}{\sqrt{N}} < x < \frac{1}{\sqrt{N}}.$$  

For any $N > 0$, let $\delta = \frac{1}{\sqrt{N}}$. Then, if $x \in (-\delta, \delta)$ and $x \neq 0$, $\frac{1}{x^2} > N$. Therefore $\lim_{x \to 0} \frac{1}{x^2} = \infty$. 