1. Find the horizontal asymptotes of the following functions, if they have any:

(a) \( f(x) = \frac{\sin\left(\frac{1}{x}\right)}{\cos\left(\frac{1}{x}\right) - 1}. \)

**Solution:** To calculate the horizontal asymptotes, we want to calculate \( \lim_{x \to \infty} f(x) \) and \( \lim_{x \to -\infty} f(x) \). Since \( \frac{1}{x} \) goes to 0 as \( x \) goes to \( \pm \infty \) and trigonometric functions are continuous, the numerator will go to \( \sin(0) = 0 \) and the denominator will go to \( 1 - \cos(0) = 0 \) as \( x \) goes to infinity. Since \( -\sin(x) = \frac{d}{dx} (\cos(x) - 1) \) is nonzero on \( (-\frac{\pi}{2}, \frac{\pi}{2}) \) except at 0, we can use L’Hospital’s Rule to evaluate the limits. So,

\[
\lim_{x \to \infty} \frac{\sin\left(\frac{1}{x}\right)}{\cos\left(\frac{1}{x}\right) - 1} = \lim_{x \to \infty} \frac{\cos\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2}}{-\sin\left(\frac{1}{x}\right) \cdot \frac{-1}{x^2}} = \lim_{x \to \infty} \frac{\cos\left(\frac{1}{x}\right)}{\sin\left(\frac{1}{x}\right)} = -\infty.
\]

Similarly, \( \lim_{x \to -\infty} f(x) = \infty \), and so we see that the function does not have any horizontal asymptotes.

(b) \( f(x) = x^2 e^x \)

**Hint:** First show that \( \lim_{x \to \infty} f(x) = \infty \) (you do not need to use L’Hospital’s Rule for this). Then, for \( \lim_{x \to -\infty} f(x) \), rewrite \( f \) as \( f(x) = \frac{x^2}{e^x} \) so that both the numerator and denominator go to infinity as \( x \) goes to negative infinity. Use L’Hospital’s Rule to show that the function has a horizontal asymptote at \( y = 0 \).

(c) \( f(x) = \frac{x \log(x^2)}{e^x} \)

**Hint:** Use L’Hospital’s Rule to show that both \( \lim_{x \to \infty} f(x) = 0 \) and \( \lim_{x \to -\infty} f(x) = 0 \).

(d) \[
f(x) = \begin{cases} 
(1 + \frac{1}{x})^x & \text{if } x > 0 \\
0 & \text{otherwise}
\end{cases}
\]

**Hint:** First show that \( \lim_{x \to -\infty} f(x) = 0 \). Next, use L’Hospital’s Rule to show that \( \lim_{x \to \infty} x \log\left(1 + \frac{1}{x}\right) = 1 \). Use this calculation to show that the function also has a horizontal asymptote at \( y = e \).
2. Find the following limits:

(a) \( \lim_{x \to 0} \frac{\cos(x) - \cos(2x)}{e^x - x - 1} \).

**Solution:** Using L'Hospital’s Rule (twice!) we get

\[
\lim_{x \to 0} \frac{\cos(x) - \cos(2x)}{e^x - x - 1} = \lim_{x \to 0} \frac{-\sin(x) + 2\sin(2x)}{e^x - 1} = \lim_{x \to 0} \frac{-\cos(x) + 4\cos(2x)}{e^x} = 3.
\]

(b) \( \lim_{x \to 0} \left( \frac{1}{x} - \csc(x) \right) \).

**Solution:** First we rewrite \( \frac{1}{x} - \csc(x) = \frac{1}{x} - \frac{1}{\sin(x)} = \frac{\sin(x) - x}{x \sin(x)} \). Since both \( \sin(x) - x \) and \( x \sin(x) \) go to zero as \( x \) goes to zero, we can use L'Hospital’s Rule to compute the limit.

\[
\lim_{x \to 0} \left( \frac{1}{x} - \csc(x) \right) = \lim_{x \to 0} \frac{\sin(x) - x}{x \sin(x)} = \lim_{x \to 0} \frac{\cos(x) - 1}{\sin(x) + x \cos(x)} = \lim_{x \to 0} \frac{-\sin(x)}{2 \cos(x) - x \sin(x)} = 0
\]

(c) \( \lim_{x \to \infty} \left( \frac{2x - 3}{2x + 5} \right)^{2x+1} \).

**Solution:** First we evaluate \( \lim_{x \to \infty} (2x + 1) \log \left( \frac{2x - 3}{2x + 5} \right) = \lim_{x \to \infty} \frac{\log \left( \frac{2x - 3}{2x + 5} \right)}{\frac{1}{2x + 1}} \) using L'Hospital’s Rule.

\[
\lim_{x \to \infty} \frac{\log \left( \frac{2x - 3}{2x + 5} \right)}{\frac{1}{2x + 1}} = \lim_{x \to \infty} \frac{1}{2x + 1} \cdot \frac{2(2x + 5) - 2(2x - 3)}{(2x - 3)^2} \frac{-2}{(2x + 1)^2} = \lim_{x \to \infty} \frac{-8(2x + 1)^2}{(2x - 3)(2x + 5)} = \lim_{x \to \infty} \frac{-8x^2(2 + \frac{1}{x})^2}{x^2(2 - \frac{3}{x})(2 + \frac{5}{x})} = -8.
\]

Hence, \( \lim_{x \to \infty} \left( \frac{2x - 3}{2x + 5} \right)^{2x+1} = e^{-8} \).
3. Let $f$ be twice differentiable. What is $\lim_{h \to 0} \frac{f(x+h)-2f(x)+f(x-h)}{h^2}$?

**Hint:** Use L'Hopital’s Rule to show that this limit is equal to

$$\lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h}$$

(remember what you’re differentiating with respect to!). Then, rewrite this limit as

$$\frac{1}{2} \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} + \frac{f'(x-h) - f'(x)}{-h}.$$ 

Use this to show that the original limit is equal to $f''(x)$. Note that you cannot use L’Hospital’s Rule twice because we are *not* told that the second derivative is continuous.

4. Explain why L’Hospital’s Rule cannot be used to evaluate $\lim_{x \to 0} \frac{\sin(x)}{x}$.

**Solution:** Recall that part of showing that $\frac{d}{dx} \sin(x) = \cos(x)$ was showing that $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ using a geometric argument. We cannot use L’Hospital’s Rule to evaluate the limit because we need to know the derivative of $\sin(x)$ in order to use L’Hospital’s Rule, but we found the derivative by evaluating the limit! That is, using L’Hospital’s Rule to evaluate the limit would be a circular argument; we would use the calculation of the limit to calculate the limit!