1. Show that the function \( f(x) = 1 + 2x + x^3 + 4x^5 \) has exactly one real root.

**Solution:** First note that \( f \) is a polynomial, so it’s continuous everywhere. So, since \( f(0) = 1 \) and \( f(-1) = -6 \), it follows from the IVT that \( f \) has at least one real root in \((-1, 0)\).

We’ll show that it can’t have more than one real root by a method called *proof by contradiction*. The way proof by contradiction works is that you assume that what you’re trying to show is not true, and then show that the consequences of that assumption are impossible, which tells us that the initial assumption cannot be true.

Since we want to show that \( f \) can have no more than one real root, we’ll assume that this isn’t true. That is, we assume that there are two numbers \( a, b \) such that \( a < b \), but \( f(a) = 0 = f(b) \). Then, since \( f \) is a polynomial, it is continuous on \([a, b]\) and differentiable on \((a, b)\), so by the Mean Value Theorem we get that there is a number \( c \in (a, b) \) for which

\[
f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{0 - 0}{b - a} = 0.
\]

However, \( f'(x) = 20x^4 + 3x^2 + 2 > 0 \) for all \( x \) (since \( x^4 \geq 0 \) and \( x^2 \geq 0 \) for all \( x \)), so it is impossible for \( f'(c) = 0 \). Therefore, we see that our initial assumption that \( f \) has at least two real roots cannot be valid, and so \( f \) must have exactly one real root.

2. Show that a polynomial of degree 3 has at most 3 distinct real roots. (Note that the on the handout, the word “distinct” was not in the question. The statement is still true without this added assumption, but I had intended for you to solve this simpler version of the problem).

**Solution:** Again, we’ll use the method of proof by contradiction to solve this problem.

Let

\[
p(x) = Ax^3 + Bx^2 + Cx + D
\]

be any polynomial of degree 3 and suppose it has four distinct real roots. That is, suppose there are numbers \( x_1 < x_2 < x_3 < x_4 \) for which \( p(x_1) = p(x_2) = p(x_3) = p(x_4) = 0 \). Since \( p \) is a polynomial, it is continuous and differentiable everywhere. So, by the Mean Value Theorem, we must have that there is a number \( c_1 \in (x_1, x_2) \) for which

\[
p'(c_1) = \frac{p(x_2) - p(x_1)}{x_2 - x_1} = \frac{0 - 0}{x_2 - x_1} = 0.
\]
Similarly, there are numbers \( c_2 \in (x_2, x_3) \) and \( c_3 \in (x_3, x_4) \) for which \( p'(c_2) = p'(c_3) = 0 \). However, \( p'(x) = 3Ax^2 + 2Bx + C \), which is a quadratic polynomial, so we know from the quadratic formula that \( p' \) can have at most two real roots, not three. We can therefore conclude that \( p \) cannot have 4 distinct real roots.

3. A number \( a \) is called a fixed point of a function \( f \) if \( f(a) = a \). Prove that if \( f \) is differentiable and \( f'(x) \neq 1 \) for all \( x \), then \( f \) can have at most one fixed point. (Note that I’ve explicitly added the assumption that \( f \) is differentiable to make the problem less ambiguous).

**Solution:** It turns out that the method of proof by contradiction is actually quite useful in mathematics (even when you’re doing research in mathematics). We’ll use it again to solve this problem.

Assume that such a function has two fixed points. That is, assume that there are two numbers \( a < b \) for which \( f(a) = a \) and \( f(b) = b \). Then, since we’re told that \( f \) is differentiable, it is continuous and we can use the Mean Value Theorem to conclude that there is a number \( c \in (a, b) \) for which

\[
    f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{b - a}{b - a} = 1.
\]

(Here we’ve used the fact that \( a \) and \( b \) are fixed points of \( f \) to simplify the numerator). This is clearly impossible since we’re told that \( f'(x) \neq 1 \) for all \( x \). So, our initial assumption cannot be true, and we can conclude that \( f \) can have at most one fixed point.

4. Prove that \( \sqrt{1 + x} < 1 + \frac{x}{2} \) for all \( x > 0 \).

**Solution:** There are a few ways to do this problem. Here’s one that involves the Mean Value Theorem.

Let \( f(x) = \sqrt{1 + x} \). Then \( f \) is continuous on \([0, \infty)\) and differentiable on \((0, \infty)\), so we can apply the Mean Value Theorem to get that, for any \( x > 0 \), there is a number \( c \in (0, x) \) for which

\[
    f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{\sqrt{1 + x} - 1}{x}.
\]

However, \( f'(c) = \frac{1}{2\sqrt{1 + c}} < \frac{1}{2} \) (since \( \sqrt{1 + c} > 1 \) for \( c > 0 \)). So we get that

\[
    \frac{1}{2} > \frac{\sqrt{1 + x} - 1}{x},
\]
which can be rearranged to give
\[ \sqrt{1 + x} < \frac{x}{2} + 1. \]

5. Let

\[ f(x) = \begin{cases} 
  x + 2x^2 \sin \left( \frac{1}{x} \right), & \text{if } x \neq 0 \\
  0, & \text{if } x = 0 .
\end{cases} \]

Use the definition of derivative to show that \( f'(0) = 1 \) but then show that \( f \) is not increasing on any interval \((-\delta, \delta)\). Is \( f' \) continuous at \( x = 0 \)?

**Solution:** By definition,
\[
f'(0) = \lim_{h \to 0} \frac{f(0 + h) - f(0)}{h} = \lim_{h \to 0} \frac{h + 2h^2 \sin \left( \frac{1}{h} \right) - 0}{h} = \lim_{h \to 0} 1 + 2h \sin \left( \frac{1}{h} \right) = 1.
\]

Note that we’ve used the fact that \( \lim_{h \to 0} h \sin \left( \frac{1}{h} \right) = 0 \). To see that this is true, notice that \(-1 \leq \sin \left( \frac{1}{h} \right) \leq 1 \) for all \( h \neq 0 \), so \(-h \leq h \sin \left( \frac{1}{h} \right) \leq h \) for all \( h \neq 0 \). Taking the limit as \( h \) goes to 0 of each side of this inequality tells us that \( 0 \leq \lim_{h \to 0} h \sin \left( \frac{1}{h} \right) \leq 0 \), from which we can conclude that the \( \lim_{h \to 0} h \sin \left( \frac{1}{h} \right) = 0 \).

Now, if \( x \neq 0 \), then
\[
f'(x) = 1 + 4x \sin \left( \frac{1}{x} \right) + 2x^2 \cos \left( \frac{1}{x} \right) \cdot \frac{-1}{x^2} = 1 + 4x \sin \left( \frac{1}{x} \right) - 2 \cos \left( \frac{1}{x} \right).
\]

For any \( \delta > 0 \), we can find \( n \in \mathbb{Z} \) sufficiently large so that \( \frac{1}{2n\pi} \in (-\delta, \delta) \) and \( \frac{1}{(2n+1)\pi} \in (-\delta, \delta) \) (taking \( n \) to be any integer larger than \( \frac{1}{2\pi\delta} \) will work), and \( f' \left( \frac{1}{2n\pi} \right) = -1 \) and \( f' \left( \frac{1}{(2n+1)\pi} \right) = 3 \). This means that in any interval \((-\delta, \delta)\), there are points at which the function is increasing and points at which it is decreasing.

The derivative \( f' \) is not continuous at \( x = 0 \) because \( \lim_{x \to 0} f'(x) \neq 1 \) (in fact, the limit does not exist). To see this, by the preceding paragraph, for any \( \delta > 0 \) we can find \( n \in \mathbb{Z} \) sufficiently large so that \( \frac{1}{(2n+1)\pi} \in (-\delta, \delta) \) and \( f' \left( \frac{1}{(2n+1)\pi} \right) = -1 \). This means that we cannot find a \( \delta > 0 \) so that \( f'(x) \) is within \( \frac{1}{2} \) of 1 for all \( x \in (-\delta, \delta) \) since \(-1\) is not within \( \frac{1}{2} \) of 1.
6. Determine where the functions are concave up, and where they are concave down.

(a) \( f(x) = x^4 \)

Solution: Since \( f''(x) = 12x^2 \), we see that \( f''(x) = 0 \) if and only if \( x = 0 \). However, since \( f''(x) > 0 \) for every \( x \neq 0 \), \( f \) does not change concavity at \( x = 0 \), and we see that \( f \) is concave up everywhere. (Note that this means \( x = 0 \) is NOT an inflection point of \( f \)).

(b) \( f(x) = x^2 + 2x + 1 \)

Solution: Since \( f''(x) = 2 > 0 \), we again see that \( f \) is concave up everywhere.

(c) \( f(x) = \frac{\log(x^2)}{x} \).

Solution: The first derivative is
\[
f'(x) = \frac{2}{x} \cdot \frac{1}{x} + \log(x^2) \cdot \frac{-1}{x^2} = \frac{2 - \log(x^2)}{x^2}.
\]
So the second derivative is
\[
f''(x) = \frac{-2}{x} \cdot \frac{1}{x^2} + (2 - \log(x^2)) \cdot \frac{-2}{x^3} = \frac{-6 + 2\log(x^2)}{x^3}.
\]
From this we see that \( f''(x) = 0 \) precisely when \( x = \pm e^{3/2} \), and \( f''(x) \) does not exist when \( x = 0 \).
For \( x \in (-\infty, -e^{3/2}) \), \( 2 \log(x^2) > 6 \) and \( x^3 < 0 \), so \( f''(x) < 0 \), and we see that \( f \) is concave down.
For \( x \in (-e^{3/2}, 0) \), \( 2 \log(x^2) < 6 \) and \( x^3 < 0 \), so \( f''(x) > 0 \), and we see that \( f \) is concave up.
For \( x \in (0, e^{3/2}) \), \( 2 \log(x^2) < 6 \) and \( x^3 > 0 \), so \( f''(x) < 0 \), and we see that \( f \) is concave down.
For \( x \in (e^{3/2}, \infty) \), \( 2 \log(x^2) > 6 \) and \( x^3 > 0 \), so \( f''(x) > 0 \), and we see that \( f \) is concave up.
So, \( f \) is concave up on \((-e^{3/2}, 0) \) and \((e^{3/2}, \infty) \), and concave down on \((-\infty, -e^{3/2}) \) and \((0, e^{3/2}) \). (Note that this means that \( x = -e^{3/2}, 0, e^{3/2} \) are all inflection points of \( f \)).