(1) In this problem, we will prove Rolle’s Theorem.

**Theorem.** (Rolle’s Theorem) Let \( f \) be continuous on \([a, b]\), differentiable on \((a, b)\) and such that \( f(a) = f(b) \). Then there is a number \( c \in (a, b) \) for which \( f'(c) = 0 \).

(a) First, \( f \) has a global maximum and a global minimum on the interval \([a, b]\). Justify this statement.

**Solution:** Since \( f \) is continuous on \([a, b]\), the Extreme Value Theorem tells us that \( f \) attains both a global maximum and a global minimum on \([a, b]\).

Note: This means that there are two cases: either \( f \) has both its global maximum and global minimum (only) at the endpoints of the interval \([a, b]\), or it has (at least) one of its global extrema in \((a, b)\).

(b) In the case where both of the global extrema occur at the endpoints of the interval, what can you conclude about the function? What can you conclude about \( f'(x) \) for every \( x \in (a, b) \)? (Hint: Remember that we are assuming that \( f(a) = f(b) \).

**Solution:** If both of the global extrema occur at the endpoints, then either \( f \) has a global minimum at \( x = a \) and global maximum at \( x = b \), or vice versa. So, since we’re assuming that \( f(a) = f(b) \), the global maximum must be equal to the global minimum and it must be that \( f \) is constant on \([a, b]\). Since the derivative of a constant is 0, \( f'(x) = 0 \) for all \( x \in [a, b] \).

(c) In the case where \( f \) has (at least) one of its global extrema in \((a, b)\), conclude that the statement of Rolle’s Theorem is true by using Fermat’s Very Little Theorem.

**Solution:** Suppose \( f \) attains its global maximum at some point \( c \in (a, b) \) (the case where \( f \) attains its global minimum at a point in \((a, b)\) is similar). Let \( \delta > 0 \) be the smaller of the two numbers \( \frac{1}{2}(c-a) \) and \( \frac{1}{2}(b-c) \) (so \( \delta \) is half of the distance between \( c \) and the endpoint which is closest to) so that the interval \((c - \delta, c + \delta)\) is contained in the interval \((a, b)\). Then \( f(c) \) is also a local minimum of \( f \), because \( f(x) \leq f(c) \) for all \( x \in (c - \delta, c + \delta) \). So since \( f \) is differentiable on \((a, b)\), by Fermat’s Very Little Theorem, \( f'(c) = 0 \).

(2) In this problem, we will use Rolle’s Theorem to prove the Mean Value Theorem.

**Theorem.** (MVT) Let \( f \) be continuous on \([a, b]\), differentiable on \((a, b)\). Then there is a number \( c \in (a, b) \) for which \( f'(c) = \frac{f(b) - f(a)}{b-a} \).

To prove the MVT, we will apply Rolle’s Theorem to the function which describes the difference between the value of the function \( f(x) \) and the value of the linear function joining the points \((a, f(a))\) and \((b, f(b))\).

(a) Find the equation of the line which passes through the points \((a, f(a))\) and \((b, f(b))\).

**Hint:** The slope of the line through the points \((a, f(a))\) and \((b, f(b))\) is \( \frac{f(b) - f(a)}{b-a} \). Use the point-slope formula for a line to find its equation.
(b) Use this to find an equation for the function describing the difference between the value of the function \( f(x) \) and the value of the linear function joining the points \((a, f(a))\) and \((b, f(b))\).

**Hint:** Let \( l(x) \) be the linear function joining the points \((a, f(a))\) and \((b, f(b))\) (from (a)). Consider the function \( g(x) = f(x) - l(x) \).

(c) Argue that you can apply Rolle’s Theorem to this function to finish the proof of the MVT.

**Hint:** Show that \( g(a) = g(b) \). Is the function \( g \) continuous on \([a, b]\)? Is it differentiable on \((a, b)\)?

(3) State whether the following statements are true or false. If the statement is true, provide justification. If it is false, provide a counterexample.

(a) Every function defined on the interval \([0, 1]\) has a maximum on the interval \([0, 1]\).

**Solution:** This statement is false! Remember that the Extreme Value Theorem only holds if the function is continuous, and we are not assuming that. This means that to find a counterexample, we should look at discontinuous functions. Consider the function

\[
f(x) = \begin{cases} 
  x & \text{if } x \neq 1 \\
  0 & \text{if } x = 1 .
\end{cases}
\]

Then \( f \) does not contain a maximum on \([0, 1]\) because it never “reaches” its maximum value. No matter how close \( x \) is to 1 (but not equal to 1), the number \( x + \frac{1-x^2}{2} < 1 \) is always closer to 1 and so \( f(x) < f\left(x + \frac{1-x^2}{2}\right) \).

(b) Suppose \( f \) is continuous on \([a, b]\) and differentiable on \((a, b)\). If there is a number \( c \in (a, b) \) for which \( f'(c) = 0 \), then \( f(a) = f(b) \).

**Solution:** This is also false! Let \( a = -1, b = 1 \) and \( f(x) = x^3 \). Then \( f'(0) = 0 \) (and \( 0 \in (-1, 1) \)), but \( f(-1) = -1 \neq 1 = f(1) \).

(c) It is possible to find a differentiable function \( f \) such that \( f(0) = -1, f(2) = 4, f'(x) \leq 2 \) for all \( x \).

**Solution:** Again, this is false! If \( f \) is differentiable (everywhere), then it is also continuous (everywhere), so we can use the Mean Value Theorem on any interval we want, in particular, the interval \([0, 2]\). The MVT tells us that there is a number \( c \in (0, 2) \) for which

\[
f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{4 - (-1)}{2} = \frac{5}{2} > 2,
\]

so it is not possible for \( f'(x) \leq 2 \) for all \( x \).

(4) If \( f(1) = 10 \) and \( f'(x) \geq 2 \) for all \( x \), how small can \( f(4) \) be?

**Hint:** Use the Mean Value Theorem and the fact that \( f'(x) \geq 2 \) for all \( x \) to show that \( f(4) \geq 16 \).

(5) Find all the local extrema of the function \( f(x) = x^2 \log(x) - 13x^2 \).

**Hint:** Note that the domain of this function is \( \{ x \in \mathbb{R} \mid x > 0 \} \), so we’re looking for local extrema on \((0, \infty)\). Show that the function has only one critical point at \( x = e^{25/2} \). Is the function increasing or decreasing to the left of \( x = e^{25/2} \)? What about to the right of \( x = e^{25/2} \)? Does this mean it should have a local maximum or a local minimum at \( x = e^{25/2} \)?