1. Sketch the graph of a function which satisfies the following criteria.
   - The domain of \( f(x) \) is \( \{ x \neq 2 \} \).
   - \( \lim_{x \to 2^-} f(x) = \infty \), \( \lim_{x \to 2^+} f(x) = -\infty \), and \( \lim_{x \to 4^-} f(x) = -\infty \).
   - \( \lim_{x \to \infty} f(x) = -1 \).
   - \( f'(x) > 0 \) for \( x \in (-\infty, 0) \cup (1, 2) \cup (2, 3) \).
   - \( f'(x) < 0 \) for \( x \in (0, 1) \cup (3, 4) \cup (4, \infty) \).
   - \( f''(x) > 0 \) for \( x \in (0.5, 2) \cup (4, \infty) \).
   - \( f''(x) < 0 \) for \( x \in (-\infty, 0.5) \cup (2, 4) \).

Solution:

2. Sketch the graphs of the following functions.
   (a) \( f(x) = \frac{x^2 - 4}{x^2 - 1} \).

Answer:
(b) \( f(x) = xe^{-x} \).

Answer:

(c) \( f(x) = (x^2 - 1)^{2/3} \).

Answer:
(d) \( f(x) = x \log(|x|) \).

Answer:

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\begin{align*}
\text{(a) Sketch the graph of } f(x) & = x^{k-1}e^{-x/\theta} / \theta^k \Gamma(k), \\
\text{where } \theta > 0, k > 3 \text{ are constants and } \Gamma(k) \text{ is the Gamma function evaluated at } k. \\
\text{The Gamma distribution can be used to model the monthly rainfall in Vancouver, i.e. the area under the graph of } f(x) \text{ between } x = a \text{ and } x = b \text{ gives the probability that there will be between } a \text{ millimetres and } b \text{ millimetres of rain in a given month.} \\
\text{Given that } f'(x) = \frac{1}{\theta^{k+1} \Gamma(k)} x^{k-2}e^{-x/\theta} \cdot ((k-1)\theta - x), \\\n\text{and } f''(x) = \frac{1}{\theta^{k+2} \Gamma(k)} x^{k-3}e^{-x/\theta} \cdot \left( x - (k-1) \left( 1 - \sqrt{1 - \frac{k-2}{k-1}} \right) \right) \left( x - (k-1) \left( 1 + \sqrt{1 - \frac{k-2}{k-1}} \right) \right), \\
\text{Sketch the graph of } f(x). \\
\text{Hint: The function has critical points } x = 0, (k-1)\theta, \text{ and the function only makes physical sense here when } x \geq 0. \text{ The inflection points are } x_1^* = \theta(k-1) \left( 1 - \sqrt{1 - \frac{k-2}{k-1}} \right) \text{ and } x_2^* = \theta(k-1) \left( 1 + \sqrt{1 - \frac{k-2}{k-1}} \right). \text{ Since } \sqrt{1 - \frac{k-2}{k-1}} > 0, \text{ it follows that } x_1^* < \theta(k-1) < x_2^*. \text{ The graph should look something like}
\end{align*}
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(b) Is it more likely that there will be between $\theta(k - 1)$ and $\theta(k - 1) + 1$ millimetres or $\theta(k - 1) + 1$ and $\theta(k - 1) + 2$ millimetres of rain in a December? (Hint: Use the graph from (a).)

Solution: Since $f(x)$ has a global maximum at $x = \theta(k - 1)$ and is decreasing for $x > \theta(k - 1)$ (by (a)), it follows that the area under the curve $y = f(x)$ between $x = \theta(k - 1)$ and $x = \theta(k - 1) + 1$ will be greater than the area under the curve between $x = \theta(k - 1) + 1$ and $x = \theta(k - 1) + 2$ (note that both areas have “base” 1).