1. Which of the following series converge? Justify your claims.

(a) \( \sum_{n=1}^{\infty} 10^{10^n} \left( 1 - \frac{1}{10^{10^n}} \right)^n \)

**Solution:** This series converges as it a geometric series with \( a = 10^{10^n} \) and \( r = 1 - \frac{1}{10^{10^n}} < 1 \).

(b) \( \sum_{n=0}^{\infty} \left( \frac{4}{3} \right)^n \)

**Solution:** This series diverges as it is a geometric series with \( a = 1 \) and \( r = \frac{4}{3} > 1 \).

(c) \( 1 - 5 - \frac{9}{5} + 2 + \pi + \left( \frac{2}{3} \right)^{12} + \left( \frac{2}{3} \right)^{13} + \left( \frac{2}{3} \right)^{14} + \cdots \)

**Solution:** This series converges. To see this, notice that we can rewrite the series as

\[
1 - 5 - \frac{9}{5} + 2 + \pi + \left( \frac{2}{3} \right)^{12} + \left( \frac{2}{3} \right)^{13} + \left( \frac{2}{3} \right)^{14} + \cdots = 1 - 5 - \frac{9}{5} + 2 + \pi + \left( \frac{2}{3} \right)^{12} \sum_{n=0}^{\infty} \left( \frac{2}{3} \right)^n .
\]

From this we can see that the series is just the sum of a convergent geometric series (with \( a = \left( \frac{2}{3} \right)^{12} \) and \( r = \frac{2}{3} < 1 \)) and a number \((1 - 5 - \frac{9}{5} + 2 - \pi)\), and so the series must converge.

(d) \( \sum_{n=1}^{\infty} \frac{1}{n} (0.5)^{n-1} \)

**Solution:** This series converges by the Comparison Test since for \( n \geq 1 \), \( \frac{1}{n} (0.5)^{n-1} \leq (0.5)^{n-1} \), and \( \sum_{n=1}^{\infty} (0.5)^{n-1} \) is a convergent geometric series (with \( a = 1 \) and \( r = 0.5 \)).
2. Prove that (possibly infinite) decimal expressions represent real numbers; that is, prove that
\[ b_k b_{k-1} \cdots b_1 b_0 . a_1 a_2 a_3 \cdots \]
is a convergent series.

**Hint:** The decimal expansion
\[ b_k b_{k-1} \cdots b_1 b_0 . a_1 a_2 a_3 \cdots, \]
represents the number given by the series
\[ b_k \cdot 10^k + b_{k-1} \cdot 10^{k-1} + \cdots + b_1 \cdot 10^1 + b_0 \cdot 10^0 + \sum_{n=1}^{\infty} a_n \cdot \left( \frac{1}{10} \right)^n. \]

To show that the series converges, use the Comparison Test to show that
\[ \sum_{n=1}^{\infty} a_n \cdot \left( \frac{1}{10} \right)^n \]
converges. (Hint: how large can \( a_n \) be?)

3. The *Comparison Test* states that if \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \) are series whose terms are (eventually) all non-negative and \( a_n \leq b_n \) (eventually), then

(i) if \( \sum_{n=1}^{\infty} b_n \) converges, then \( \sum_{n=1}^{\infty} a_n \) converges.

(ii) if \( \sum_{n=1}^{\infty} a_n \) diverges, then \( \sum_{n=1}^{\infty} b_n \) diverges.

(a) Prove (i) by showing that the sequence of partial sums \( s_n = \sum_{n=1}^{\infty} a_n \) is (eventually) monotone increasing and bounded and applying the Monotone Convergence Theorem.

**Solution:** We will show that (i) is true when \( a_n, b_n \geq 0 \) for all \( n \) and \( a_n < b_n \) for all \( n \). The more general case is a consequence of this case since this will show that
\[ \sum_{n=N}^{\infty} a_n \]
converges for some \( N \), and the first \( N-1 \) terms of the series cannot affect its convergence.

Since \( a_n \geq 0 \) for all \( n \), \( s_n = s_{n-1} + a_n \geq s_{n-1} \) for all \( n > 1 \) which means that the sequence of partial sums is monotone. Now, since \( a_n \leq b_n \) for all \( n \), and \( b_n \geq 0 \) for all \( n \),
\[ s_N = \sum_{n=1}^{N} a_n \leq \sum_{n=1}^{N} b_n \leq \sum_{n=1}^{\infty} b_n. \]

Now, if \( \sum_{n=1}^{\infty} b_n \) converges, then \( \sum_{n=1}^{\infty} b_n \) is just a finite number and therefore provides an upper bound for the sequence of partial sums. It now follows from the MCT that \( \sum_{n=1}^{\infty} a_n \) converges.
(b) Prove (ii) by first showing that, since \( \sum_{n=1}^{\infty} a_n \) diverges and \( a_n \) is eventually non-negative, \( \sum_{n=1}^{\infty} a_n \) must diverge to infinity. Use this and the definition of divergence to infinity to show that \( \sum_{n=1}^{\infty} b_n \) also diverges.

Solution: Again, we will show that (ii) is true when \( a_n, b_n \geq 0 \) for all \( n \), and \( a_n \leq b_n \) for all \( n \). Since \( a_n \geq 0 \), \( \sum_{n=1}^{\infty} a_n \) diverges to infinity. This means that, for any number \( M > 0 \), there is a number \( N \) such that \( s_n > M \) for all \( n > N \). However, since \( b_n > a_n \) for all \( n \),

\[
\sum_{k=1}^{n} b_k \geq \sum_{k=1}^{n} a_k = s_n > M
\]

for all \( n > N \). That is, the partial sums of \( \sum_{n=1}^{\infty} b_n \) can be made arbitrarily large, and so \( \sum_{n=1}^{\infty} b_n \) must also diverge to infinity.