Practice Problems:

1. Find the Maclaurin series for the following functions. What is the radius of convergence? (Hint: For (d)-(g) you should not have to calculate \( f^{(n)}(0) \) – you can use power series methods to find the series).

(a) \( e^x \)

**Solution:** For all \( x \),

\[
e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \ldots
\]

(b) \( \cos(x) \)

**Solution:** For all \( x \),

\[
\cos(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \ldots
\]

(c) \( \sin(x) \)

**Solution:** For all \( x \),

\[
\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{6} + \frac{x^5}{120} - \ldots
\]

(d) \( \log(1 + x) \)

**Solution:** For \( |x| < 1 \),

\[
\log(1 + x) = \int_{0}^{x} \frac{1}{1+t} dt = \int_{0}^{x} \left( \sum_{n=0}^{\infty} (-1)^n t^n \right) dt = \sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1}.
\]

(e) \( \frac{1}{1-x} \)

**Solution:** For \( |x| < 1 \),

\[
\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n.
\]
(f) \( \frac{1}{(1+x)^2} \)

Solution: For \(|x| < 1\),

\[
\frac{1}{(1+x)^2} = \frac{d}{dx} \left( \frac{-1}{1+x} \right) = -\sum_{n=0}^{\infty} (-1)^n x^n = \sum_{n=0}^{\infty} (-1)^{n+1} x^n.
\]

(g) \( \frac{1}{6+x^2} \)

Solution: For \(|x| < 6\),

\[
\frac{1}{6+x^2} = \frac{1}{6} \cdot \frac{1}{1 + \frac{x^2}{6}} = \frac{1}{6} \sum_{n=0}^{\infty} (-1)^n \left( \frac{x^2}{6} \right)^n = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{6^{n+1}}.
\]

2. Find the Maclaurin series for \(\sin(x^2)\). Where does it converge? What about \(\sin(x^2)/x\)?

Hint: For all \(x\),

\[
\sin(x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n x^{4n+2}.
\]

Note that the series obtained by dividing the Maclaurin series for \(\sin(x^2)\) by \(x\) is NOT the Maclaurin series for \(f(x) = \sin(x^2)/x\), since \(f(0), f'(0), f''(0), \ldots\) are not defined. Instead, one can show that it is the Maclaurin series for the function

\[
g(x) = \begin{cases} 
\sin(x^2), & \text{if } x \neq 0 \\
0, & \text{if } x = 0
\end{cases}.
\]

3. If \(f(x) = \cos(3x)\), what is \(f^{(105)}(0)\)? What about \(f^{(106)}(0)\)?

Solution: We know that, for all \(x\),

\[
\cos(3x) = \sum_{n=0}^{\infty} \frac{(-1)^n (3x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{2n}}{(2n)!},
\]

and that this is the Maclaurin series for \(f(x) = \cos(3x)\), i.e.

\[
\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{(-1)^n 3^{2n} x^{2n}}{(2n)!}.
\]

Hence \(f^{(105)}(0) = 0\), and \(f^{(106)}(0) = 106! \cdot (-1)^{53} \frac{3^{106}}{106!} = -3^{106} \).
4. Without using l’Hospital’s Rule, evaluate

$$\lim_{x \to 0} \frac{(e^{2x} - 1) \log(1 + x^2)}{(1 - \cos(3x))^2}.$$ 

**Hint:** Expand the functions using Taylor series and show that the limit is equal to 0.