MATH 101 V2A
January 28th – Practice problems
Hints and Solutions

1. Integration by Parts:

(a) Evaluate \( \int_1^e \log(t) dt \).

**Solution:** Let \( u = \log(t) \) and \( dv = dt \). Then \( du = \frac{1}{t} dt \), \( v = t \) and using the method of parts gives us
\[
\int_1^e \log(t) dt = t \log(t) \bigg|_1^e - \int_1^e \frac{1}{t} dt = e - \left( t \bigg|_1^e \right) = e - (e - 1) = 1.
\]

(b) Evaluate \( \int_e^{e^2} t^2 \frac{\log(t)}{t} dt \).

**Solution:** I've just realized that this problem isn’t really doable. I hope you haven’t wasted too much time on it – if you have, you might feel better in knowing that I have also wasted a lot of time on it!!

(c) Let \( f(t) \) be continuously differentiable. Prove that \( \int_a^b f(t) f'(t) dt = \frac{1}{2} \left( f(b)^2 - f(a)^2 \right) \).

**Solution:** Let \( u = f(t) \) and \( dv = f'(t) dt \). Then \( du = f'(t) dt \), \( v = f(t) \) and the method of parts gives us that
\[
\int_a^b f(t) f'(t) dt = f(t) \cdot f(t) \bigg|_a^b - \int_a^b f(t) f'(t) dt.
\]
So
\[
2 \int_a^b f(t) f'(t) dt = f(b)f(b) - f(a)f(a),
\]
and therefore
\[
\int_a^b f(t) f'(t) dt = \frac{1}{2} \left( f(b)^2 - f(a)^2 \right).
\]
You could also do this problem using the method of substitution.

2. Partial Fractions:

(a) Evaluate \( \int_2^3 \frac{6}{t^2 - 1} dt \).

**Hint:** Factor the denominator into \( t^2 - 1 = (t - 1)(t + 1) \) and use partial fractions to get that
\[
\frac{6}{t^2 - 1} = \frac{-3}{t + 1} + \frac{3}{t - 1}
\]
Use this to get that

\[
\int_2^3 \frac{6}{t^2 - 1} \, dt = -3 \log(2) + 3 \log(3) - 3 \log(4) = 3 \log \left( \frac{3}{2} \right).
\]

(b) Evaluate \( \int_{-4}^{-3} \frac{6t^2}{t^4 - 5t^2 + 4} \, dt \).

**Hint:** Factor the denominator into \( t^4 - 5t^2 + 4 = (t + 2)(t - 2)(t + 1)(t - 1) \) and use partial fractions to get that

\[
\frac{6t^2}{t^4 - 5t^2 + 4} = \frac{2}{t - 2} + \frac{-2}{t + 2} + \frac{-1}{t + 1} + \frac{1}{t + 1}.
\]

Use this to get that

\[
\int_{-4}^{-3} \frac{6t^2}{t^4 - 5t^2 + 4} \, dt = 3 \log(5) - \log(4) + 3 \log(2) - \log(3) - 2 \log(6) = \log \left( \frac{5^3}{2 \cdot 3^3} \right).
\]

(c) Evaluate \( \int_{-1}^{1} \frac{2t^3 + 11t^2 + 28t + 33}{t^2 - t - 6} \, dt \).

**Hint:** Use long division to divide the polynomials and get that \( 2t^3 + 11t^2 + 28t + 33 = (t^2 - t - 6)(2t + 13) + 53t + 111 \). Use this to rewrite the integral as

\[
\int_{-1}^{1} \frac{2t^3 + 11t^2 + 28t + 33}{t^2 - t - 6} \, dt = \int_{-1}^{1} \frac{53t + 111}{t^2 - t - 6} \, dt
\]

\[
= t^2 + 13t \bigg|_{-1}^{1} + \int_{-1}^{1} \frac{53t + 111}{t^2 - t - 6} \, dt
\]

\[
= 26 + \int_{-1}^{1} \frac{53t + 111}{t^2 - t - 6} \, dt.
\]

Now use partial fractions on the remaining integral.

**Answer:** \( 26 + 54 \log(2) - \log(3) - 54 \log(4) = 26 - 54 \log(2) - \log(3) \).