1. Evaluate the following integrals, then sketch the region whose area is represented by the integral.

(a) \( \int_0^b t^2 + 1 \, dt \)

**Solution:** Since \( f(t) = t^2 + 1 \) is continuous (everywhere) and \( F(t) = \frac{1}{3} t^3 + t \) is an antiderivative of \( f(t) \), it follows from the Fundamental Theorem of Calculus that

\[
\int_0^b t^2 + 1 \, dt = \frac{1}{3} t^3 + t \bigg|_0^b = \frac{1}{3} b^3 + b.
\]

(b) \( \int_{-1}^0 2e^t \, dt \)

**Answer:** \( \int_{-1}^0 2e^t \, dt = 2 - \frac{2}{e} \).

(c) \( \int_{-\pi}^{\pi} \sin(t) \, dt \)

**Answer:** \( \int_{-\pi}^{\pi} \sin(t) \, dt = 0 \)

2. Find the area above \( y = t^2 + 3 \) and below \( y = 4t \).

**Solution:** The two curves intersect when \( 4t = t^2 + 3 \), i.e., when \( 0 = t^2 - 4t + 3 = (t - 3)(t - 1) \). Therefore, the two curves intersect when \( t = 1 \) and when \( t = 3 \). Between \( t = 1 \) and \( t = 3 \), the curve \( y = 4t \) lies
above the curve \( y = t^2 + 3 \), so the area we’re asked to calculate is given by

\[
\int_1^3 4t - (t^2 + 3)\,dt.
\]

Since \( f(t) = 4t - (t^2 + 3) \) is continuous (everywhere) and \( F(t) = 2t^2 - \frac{1}{3}t^3 - 3t \) is an antiderivative of \( f(t) \), it follows from the Fundamental Theorem of Calculus that

\[
\int_1^3 4t - (t^2 + 3)\,dt = 2t^2 - \frac{1}{3}t^3 - 3t\bigg|_1^3 = \frac{4}{3}.
\]

3. Solve the integral equation \( f(x) = 1 + 4\int_3^x f(t)\,dt \). (Hint: differentiate the equation, and then solve the resulting differential equation.)

**Solution:** By the Fundamental Theorem of Calculus, differentiating both sides of the equation yields the differential equation

\[
f'(x) = 4f(x).
\]

Note that \( e^{4x} \) is a solution to this differential equation, but it is **not** the only solution to the differential equation (and you can check that \( e^{4x} \) is not a solution to the original integral equation). However, any solution to the differential equation is of the form \( f(x) = Ae^{4x} \), where \( A \) is a constant. From the integral equation, we see that

\[
f(3) = 1 + 4\int_3^3 f(t)\,dt = 1,
\]

since \( \int_3^3 f(t)\,dt = 0 \) (intuitively, this integral represents an area with zero width). Therefore we want to choose \( A \) such that \( f(3) = 1 \), i.e., we want \( A \) such that \( Ae^{12} = 1 \). Hence \( A = e^{-12} \), and the solution to the integral equation is \( f(x) = e^{4x-12} \).