1. Solve the following differential equations.

(a) \( \frac{dy}{dx} x^2 + 2xy = \log(x) \). What if \( y(1) = 5 \)?

**Solution:** Notice that \( \frac{d}{dx} (x^2y) = 2xy + x^2 \frac{dy}{dx} \). This means that

\[
\frac{d}{dx} (x^2y) = \log(x)
\]

so, using integration by parts,

\[
x^2y = \int \log(x) = x \log(x) + x + C.
\]

Therefore

\[
y = \frac{\log(x)}{x} + \frac{1}{x} + \frac{C}{x^2}.
\]

Note that you can also solve this using an integrating factor (in this case, the integrating factor is \( x^2 \) (remember that, in this case, you have to divide the equation by \( x^2 \) before you use the formula to calculate the integrating factor), so the two methods are really the same). Now, if \( y(1) = 5 \), then \( 5 = 0 + 1 + C \), so \( C = 4 \). Therefore the solution is \( y = \frac{\log(x)}{x} + \frac{1}{x} + \frac{4}{x^2} \).

(b) \( \frac{dy}{dx} = \frac{\log(x)}{xy} \). What if \( y(1) = 2 \)?

**Solution:** If \( \frac{dy}{dx} = \frac{\log(x)}{xy} \), then \( y \frac{dy}{dx} = \frac{\log(x)}{x} \). So, using the method of substitution (\( u = \log(x) \) – remember that you must convert the integral back into being a function of \( x \) from being a function of \( u \))

\[
\frac{1}{2} y^2 = \frac{1}{2} \log(x)^2 + C,
\]

or

\[
y^2 = \log(x)^2 + D.
\]

If \( y(1) = 2 \), then

\[
4 = 0 + D,
\]

so the solution is \( y^2 = \log(x)^2 + 4 \). Note that, in this case, we can actually solve for \( y \) explicitly. Since, \( y(1) = 2 > 0 \) and \( \log(x)^2 + 4 \neq 0 \) for all \( x > 0 \), \( y = \sqrt{\log(x)^2 + 4} \).

(c) \( \frac{dy}{dx} x - 4y = x^4 \log(x) \).

**Solution:** If \( \frac{dy}{dx} x - 4y = x^4 \log(x) \), then \( \frac{dy}{dx} - 4 \frac{y}{x} = x^3 \log(x) \), so the integrating factor for this differential equation is

\[
I = e^{\int -4/x \ dx} = e^{-4 \log(x)} = e^{\log(1/x^4)} = \frac{1}{x^4}.
\]
Therefore
\[ \frac{d}{dx} \left( \frac{1}{x^4} y \right) = x^3 \log(x) \cdot \frac{1}{x^4} = \frac{\log(x)}{x}. \]

Using the method of substitution \((u = \log(x))\) we get
\[ \frac{1}{x^4} y = \frac{1}{2} \log(x)^2 + C, \]
so
\[ y = \frac{x^4 \log(x)^2}{2} + Cx^4. \]

2. Find and sketch the orthogonal trajectories of \(y = kx^2\)? (An orthogonal trajectory of a family of curves is a curve that intersects each member of the family orthogonally. Recall that two curves intersect orthogonally if the product of their slopes is equal to \(-1\) at the points at which they intersect. **Hint:** You need to find a differential equation that holds for all \(k\) simultaneously.)

**Solution:** If \(\tilde{y}\) is an orthogonal trajectory to the family of curves \(y = kx^2\), then we must have that \(\tilde{y}' \cdot y' = -1\). Since \(y' = 2kx\), this gives us that \(\tilde{y}' = -\frac{x}{2y}\). Now, we need an equation that is independent of \(k\) (otherwise the solution that we find will depend on \(k\), but we want to find a trajectory that is orthogonal to every curve of the form \(y = kx^2\)). We want \(\tilde{y}\) to be orthogonal to \(y = kx^2\) when they intersect, that is, when \(\tilde{y} = y = kx^2\). We can solve this for \(k\) to get \(k = \frac{\tilde{y}}{x^2}\). So, substituting this into our other equation gives us the differential equation
\[ \tilde{y}' = -\frac{1}{2} \frac{x}{x^2} = -\frac{x}{2\tilde{y}}. \]

So,
\[ \tilde{y}' \, dy = -\frac{x}{2} \, dx, \]
and we get that
\[ \frac{1}{2} \tilde{y}^2 = -\frac{1}{4} x^2 + C, \]
which can be rewritten as
\[ \tilde{y}^2 + \frac{x^2}{2} = D. \]

So we see that the orthogonal trajectories are ellipses.

3. Suppose 10kg of salt is mixed into 100L of water. Pure water enters at 15L/min, and the mixed tank is drained at 18L/min. How much salt remains after \(t\) minutes?

**Solution:** Let \(S(t)\) be the amount of salt remaining in the tank at time \(t\) and let \(V(t)\) be the volume of solution in the tank at time \(t\). Then \(V(t) = 100 - 3t\). Now, since no salt is being added to the tank, we have that \(S'(t)\) is equal to the rate at which the salt leaves the tank. Since the solution is leaving the tank at a rate of 18L/min and the concentration of the solution at time \(t\) will be \(\frac{S(t)}{100-3t}\), we have that
\[ \frac{dS}{dt} = \frac{S(t)}{100-3t} \cdot (-18), \]
and so
\[
\frac{1}{S}dS = -\frac{18}{100-3t}dt.
\]

Hence,
\[
\log(S(t)) = 6\log(100-3t) + C,
\]
and so (setting \(A = e^C\))
\[
S(t) = A e^{6\log(100-3t)} = A(100-3t)^6.
\]

Since \(S(0) = 10\) kg, we get that \(A = \frac{10}{100^6}\). Therefore
\[
S(t) = \frac{10}{100^6}(100-3t)^6.
\]

Note that this only holds on for \(t \in [0, \frac{100}{3}]\) since the solution is only exiting the tank at a rate of
18L/min for such \(t\) (after that the tank is empty).