Short Answer Questions

Evaluate the following integrals or state that they diverge.

1. \( \int_0^5 \frac{x}{x + 10} \, dx \)

Solution: Let \( u = x + 10 \). Then \( du = dx \) and \( x = u - 10 \) so that \( \frac{x}{x + 10} \, dx = \frac{u - 10}{u} \, du \). When \( x = 0 \), \( u = 10 \), and when \( x = 5 \), \( u = 15 \). Therefore,

\[
\int_0^5 \frac{x}{x + 10} \, dx = \int_{10}^{15} \frac{u - 10}{u} \, du = \int_{10}^{15} (1 - \frac{10}{u}) \, du = (u - 10 \ln |u|) \bigg|_{10}^{15} = 15 - 10 \ln(15) - 10 + 10 \ln(10) = 10(\ln(15) - \ln(10)).
\]

2. \( \int_3^4 \frac{1}{y^2 - 4y - 12} \, dy \)

Solution: First, note that \( y^2 - 4y - 12 = (y - 6)(y + 2) \). To use the method of partial fractions we need to find \( A \) and \( B \) such that

\[
\frac{1}{y - 6)(y + 2)} = \frac{A}{y - 6} + \frac{B}{y + 2}.
\]

Multiplying both sides by \( (y - 6)(y + 2) \) we get that

\[
1 = A(y + 2) + B(y - 6).
\]

Plugging \( y = -2 \) into the equation we see that \( 1 = B \cdot (-8) \), or \( B = -\frac{1}{8} \). Plugging \( y = 6 \) into the equation we see that \( 1 = A \cdot (8) \), or \( A = \frac{1}{8} \). Therefore

\[
\frac{1}{(y - 6)(y + 2)} = \frac{1/8}{y - 6} + \frac{-1/8}{y + 2},
\]

and so

\[
\int_3^4 \frac{1}{y^2 - 4y - 12} \, dy = \int_3^4 \frac{1/8}{y - 6} \, dy + \int_3^4 \frac{-1/8}{y + 2} \, dy = \frac{1}{8} \ln |y - 6| \bigg|_3^4 - \frac{1}{8} \ln |y + 2| \bigg|_3^4 = \frac{1}{8} (\ln(2) - \ln(3) - \ln(6) + \ln(5)).
\]
3. $\int_0^a \frac{1}{\sqrt{a^2 - x^2}} \, dx$

Solution: Note that this is an improper integral, so when evaluating at $x = a$, we're really doing this by evaluating at $t < a$ and taking the limit of the result as $x \to a^-$. Let $u = a \sin(x)$. Then $du = a \cos(x) \, dx$, and $\sqrt{a^2 - x^2} = a \cos(u)$. Furthermore, when $x = 0$, $\sin(u) = 0$, so $u = 0$. When $x = a$, $\sin(u) = 1$, so $u = \pi/2$. Therefore,

$$\int_0^a \frac{1}{\sqrt{a^2 - x^2}} \, dx = \int_0^{\pi/2} \frac{1}{a \cos(u)} \cos(u) \, du = \int_0^{\pi/2} \frac{1}{a \cos(u)} \, du = \left. \frac{\pi}{2} \right|_0^{\pi/2} = \frac{\pi}{2}$$

So, the integral converges, and the value of the integral is $\frac{\pi}{2}$.

4. $\int_2^\infty \frac{1}{x \ln(x)} \, dx$

Solution: Let $u = \ln(x)$. Then $du = \frac{1}{x} \, dx$ and so $\int \frac{1}{x} \, dx = \int du$. Furthermore, when $x = 2$, $u = \ln(2)$, and when $x = \infty$, $u = \infty$. Therefore,

$$\int_2^\infty \frac{1}{x \ln(x)} \, dx = \int_\ln(2)^{\infty} \frac{1}{u} \, du = \left. \ln |u| \right|_{\ln(2)}^{\infty} = \infty.$$

Therefore the integral diverges.

5. $\int \tan^2(u) \cos^2(u) \, du$

Solution: We have that

$$\int \tan^2(u) \cos^2(u) \, du = \int \frac{\sin^2(u)}{\cos^2(u)} \cos^2(u) \, du = \int \sin^2(u) \, du$$

$$= \int \left( \frac{1}{2} - \frac{1}{2} \cos(2u) \right) \, du$$

$$= \left. \frac{u}{2} - \frac{1}{4} \sin(2u) \right| + C$$

6. $\int \csc^4(\theta) \cos(\theta) \, d\theta$
Solution: First recall that $\csc^4(\theta) = \frac{1}{\sin^4(\theta)}$. Let $u = \sin(\theta)$. Then $du = \cos(\theta)d\theta$, and we get that

$$\int \csc^4(\theta) \cos(\theta)d\theta = \int \frac{1}{u^4} du = -\frac{1}{3u^3} + C = -\frac{1}{3\sin^3(\theta)} + C.$$ 

7. $\int \frac{x^2 + 2}{x + 2} dx$

Solution: Using long division of polynomials we get that

$$\frac{x^2 + 2}{x + 2} = x - 2 + \frac{6}{x + 2}.$$ 

So,

$$\int \frac{x^2 + 2}{x + 2} dx = \int \left(x - 2 + \frac{6}{x + 2}\right) dx = \frac{1}{2} x^2 - 2x + 6 \ln |x + 2| + C.$$ 

8. $\int t \cos(t^2) dt$

Solution: Let $u = t^2$. Then $du = 2tdt$ (or $\frac{1}{2} du = tdt$) and we get that

$$\int t \cos(t^2) dt = \int \frac{1}{2} \cos(u) du = \frac{1}{2} \sin(u) + C = \frac{1}{2} \sin(t^2) + C.$$ 

9. $\int x^{3/2} \ln(x) dx$

Solution: Let $u = \ln(x)$ and $dv = x^{3/2} dx$ so that $du = \frac{1}{x} dx$ and $v = \frac{2}{5} x^{5/2}$. Using integration by parts we have that

$$\int x^{3/2} \ln(x) dx = \frac{2}{5} x^{5/2} \ln(x) - \int \frac{2}{5} x^{5/2} \frac{1}{x} dx$$

$$= \frac{2}{5} x^{5/2} \ln(x) - \int \frac{2}{5} x^{3/2} dx$$

$$= \frac{2}{5} x^{5/2} \ln(x) - \frac{4}{25} x^{5/2} + C.$$
10. \[ \int \ln(x)\,dx \quad \text{(Hint: Think of } \ln(x) \text{ as } 1 \cdot \ln(x).) \]

**Solution:** Let \( u = \ln(x) \) and \( dv = 1\,dx \) so that \( du = \frac{1}{x}\,dx \) and \( v = x \). Using integration by parts we have that
\[
\int \ln(x)\,dx = x \ln(x) - \int \frac{1}{x}\,dx = x \ln(x) - \int dx = x \ln(x) - x + C.
\]

11. \( \int_1^2 \frac{1}{\sqrt{x-1}}\,dx \)

**Solution:** Note that this is an improper integral, so when we evaluate at \( x = 1 \), we’re really doing this by evaluating at \( t > 1 \) and taking the limit of the result as \( t \to 1^+ \). We have that
\[
\left[ \int_1^2 \frac{1}{\sqrt{x-1}}\,dx \right] = 2 \sqrt{x-1} \bigg|_1^2 = 2.
\]
So, the integral converges, and the value of the integral is 2.

**Long Answer Questions**

1. Let \( F(x) = \int_1^x \frac{1}{t^2 + 6t + 5}\,dt \). Find the equation of the line tangent to \( F(x) \) at \( x = 2 \).

**Solutions:** From the FTC we know that \( F'(x) = \frac{1}{x^2 + 6x + 5} \), so \( F'(2) = \frac{1}{22} \). Also,
\[
F(2) = \int_1^2 \frac{1}{t^2 + 6t + 5}\,dt.
\]
Now \( \frac{1}{t^2 + 6t + 5} = \frac{1}{(t+1)(t+5)} \), so to use the method of partial fractions we need to find \( A \) and \( B \) such that
\[
\frac{1}{(t+1)(t+5)} = \frac{A}{t+1} + \frac{B}{t+5}.
\]
Multiplying both sides of this equality by \( (t+1)(t+5) \) we get
\[
1 = A(t+5) + B(t+1).
\]
Plugging \( t = -1 \) into the above equality gives \( 1 = A(4) \), or \( A = 1/4 \). Plugging \( t = -5 \) into the above
equality gives 1 = B(−4), or B = −1/4. Therefore,

\[
\int_1^2 \frac{1}{(t+1)(t+5)} \, dt = \int_1^2 \left( \frac{1/4}{t+1} - \frac{1/4}{t+5} \right) \, dt
\]

\[
= \left. \left( \frac{1}{4} \ln |t+1| - \frac{1}{4} \ln |t+5| \right) \right|_1^2
\]

\[
= \frac{1}{4} \left( \ln(3) - \ln(7) - \ln(2) + \ln(6) \right)
\]

\[
= \frac{1}{4} \ln \left( \frac{9}{7} \right)
\]

(It’s not necessary to simplify to get the last equality). So \( F(2) = \frac{1}{4} \ln \left( \frac{9}{7} \right) \). Therefore, the equation of the tangent line is given by

\[
y - \frac{1}{4} \ln \left( \frac{9}{7} \right) = \frac{1}{21}(x - 2).
\]

(You do not need to rearrange the above equation to solve for \( y \)).

2. Recall that a differentiable function \( F(x) \) is said to be increasing at a point \( a \) if \( F'(a) \geq 0 \). Show that the function

\[
\int_0^x e^t \, dt
\]

is always increasing.

**Solution:** Let \( G(x) = \int_0^x e^t \, dt \). Then, from the FTC, we know that \( G'(x) = e^x \). Now, if \( F(x) = \int_0^x e^t \, dt \), then \( F(x) = G(e^x) \). So, from the chain rule we have that

\[ F'(x) = G'(e^x) \cdot e^x = e^{e^x} \cdot e^x. \]

Since \( e^x > 0 \) for all \( x \), it follows that \( F'(x) > 0 \) for all \( x \) and is therefore always increasing.

3. Find the derivative of the function \( F(x) = \int_0^x xe^t \, dt \). (Hint: \( x \) does not depend on \( t \)).

**Solution:** Let \( G(x) = \int_0^x e^t \, dt \). Then, from the FTC, we have that \( G'(x) = e^x \). Since \( x \) does not depend on \( t \) (the variable of integration), \( F(x) = \int_0^x xe^t \, dt = x \int_0^x e^t \, dt = xG(x) \). Therefore, using the product rule we have that

\[ F'(x) = \frac{d}{dx} (xG(x)) = G(x) + xG'(x) = \int_0^x e^t \, dt + xe^x = e^x \bigg|_0^x + xe^x = e^x - 1 + xe^x. \]
4. If \( f(x) \) is continuous on \([a, b]\), then the average value of \( f(x) \) on \([a, b]\) is defined to be

\[
\text{ave} = \frac{1}{b-a} \int_{a}^{b} f(x)\,dx.
\]

Find the average value of the function \( f(x) = \frac{x^3 + 4}{x^2 + 4x + 3} \) on the interval \([0, 2]\).

**Solution:** Using long division of polynomials we get that

\[
\frac{x^3 + 4}{x^2 + 4x + 3} = x - 4 + \frac{13x + 16}{x^2 + 4x + 3}.
\]

Now, note that \( x^2 + 4x + 3 = (x + 1)(x + 3) \). So, to use the method of partial fractions on the second term above we need to find \( A \) and \( B \) such that

\[
\frac{13x + 16}{(x + 1)(x + 3)} = \frac{A}{x + 1} + \frac{B}{x + 3}.
\]

Multiplying this through by \((x + 1)(x + 3)\) we get that

\[
13x + 16 = A(x + 3) + B(x + 1).
\]

Plugging \( x = -3 \) into the above equation and solving for \( B \) we get that \( B = 23/2 \). Plugging \( x = -1 \) into the above equation and solving for \( A \) we get that \( A = 3/2 \). So

\[
\frac{13x + 16}{(x + 1)(x + 3)} = \frac{3/2}{x + 1} + \frac{23/2}{x + 3}.
\]

Therefore,

\[
f_{\text{ave}} = \frac{1}{2} \int_{0}^{2} \left( x - 4 + \frac{3/2}{x + 1} + \frac{23/2}{x + 3} \right) \,dx
\]

\[
= \frac{1}{2} \left[ \frac{1}{2} x^2 - 4x + \frac{3}{2} \ln|x + 1| + \frac{23}{2} \ln|x + 3| \right]_{0}^{2}
\]

\[
= \frac{1}{2} \left( \frac{1}{2} \cdot 2^2 - 8 + \frac{3}{2} \ln(3) + \frac{23}{2} \ln(5) - 0 - \frac{3}{2} \ln(1) - \frac{23}{2} \ln(3) \right)
\]

\[
= \frac{1}{2} \left( -6 + \frac{23}{2} \ln(5) - 10 \ln(3) \right).
\]

5. Find the area of the overlapping portion of the circles \( x^2 + (y - 1)^2 = 1 \) and \( x^2 + y^2 = 1 \) that is in the first quadrant. (Note: The equation \( x^2 + (y - 1)^2 = 1 \) is the equation of the circle of radius 1 centred at the point \((0, 1)\), and the equation \( x^2 + y^2 = 1 \) is the equation of the circle of radius 1 centred at the origin.)

**Solution:** First note that the circles intersect when \((1 - (y - 1)^2) + y^2 = 1\) (solving \( x^2 + (y - 1)^2 = 1 \) for \( x^2 \) and then substituting this into the equation of the other circle). So, the circles intersect when \( 2y - 1 = 0 \), or, equivalently, when \( y = 1/2 \). Now, if \( y = 1/2 \), then substituting this into \( x^2 + y^2 = 1 \) we see that \( x = \sqrt{3}/2 \). Now, the top of the circle centred at the origin is given by \( y = \sqrt{1 - x^2} \), and
the bottom of the circle centred at the point (0, 1) is given by \( y = 1 - \sqrt{1-x^2} \). So, the area of the overlapping portion of the circles in the first quadrant is

\[
\text{area} = \int_0^{\sqrt{3}/2} \left[ (\sqrt{1-x^2}) - (1 - \sqrt{1-x^2}) \right] dx = \int_0^{\sqrt{3}/2} (2\sqrt{1-x^2} - 1) dx.
\]

Now, let \( x = \sin(u) \). Then \( dx = \cos(u) du \) and \( \cos(u) = \sqrt{1-x^2} \). Also, when \( x = 0 \), \( \sin(u) = 0 \), so \( u = 0 \), and when \( x = \sqrt{3}/2 \), \( \sin(u) = \sqrt{3}/2 \), so \( u = \pi/3 \). Using this trig substitution we get

\[
\text{area} = \int_0^{\pi/3} (2\cos(u) - 1) \cos(u) du = \int_0^{\pi/3} (2\cos^2(u) - \cos(u)) du \\
= \int_0^{\pi/3} (1 + \cos(2u) - \cos(u)) du \\
= \left( u + \frac{1}{2} \sin(2u) - \sin(u) \right) \bigg|_0^{\pi/3} \\
= \frac{\pi}{3} + \frac{\sqrt{3}}{4} - \frac{\sqrt{3}}{2} - 0 \\
= \frac{\pi}{3} - \frac{\sqrt{3}}{4}.
\]

6. Find the area between the curves \( y = e + \sin^2(\pi x) \) and \( y = xe^x \) in the first quadrant.

\[ \text{Solution:} \quad \text{The area between the two curves in the first quadrant is given by the integral} \]

\[
\text{area} = \int_0^1 [(e + \sin^2(\pi x)) - xe^x] dx = \int_0^1 \left( e + \frac{1}{2} - \frac{1}{2} \cos(2\pi x) \right) dx - \int_0^1 xe^x dx.
\]

We can calculate the first integral directly

\[
\int_0^1 \left( e + \frac{1}{2} - \frac{1}{2} \cos(2\pi x) \right) dx = \left( e \cdot x + \frac{x}{2} - \frac{1}{4\pi} \sin(2\pi x) \right) \bigg|_0^1 = e + \frac{1}{2}
\]

To calculate the second integral we will use the method of integration by parts with \( u = x \) and
\[ dv = e^x \, dx \text{.} \] Then \[ du = dx, \quad v = e^x, \] and IBP gives us
\[
\int_0^1 xe^x \, dx = xe^x \bigg|_0^1 - \int_0^1 e^x \, dx
\]
\[
= e - \left( e^1 \bigg|_0 \right)
\]
\[
= e - (e - 1) = 1.
\]

So, the area between the two curves in the first quadrant is \[ \text{area} = e - \frac{1}{2} \].

7. (a) Use sigma notation to write down a Midpoint Riemann Sum approximation of the area under the curve \( y = e^{1/t} \) between \( t = 1 \) and \( t = 2 \) with \( n = 10 \). Do not evaluate the Riemann sum.

**Solution:** For this problem \( a = 1, \quad b = 2, \quad n = 10 \) and \( f(t) = e^{1/t} \). So, \( \delta t = \frac{b-a}{n} = \frac{1}{10} \). So, the \( \text{ith} \) midpoint is \( 1 + (2i - 1)/20 \), and the approximation is given by
\[
\int_1^2 e^{1/t} \, dt \approx \sum_{i=1}^{10} e^{1/(1+2i/20)} \cdot \frac{1}{10}.
\]

(b) Use the error formula
\[
|error| \leq \frac{N(b-a)^3}{24n^2}
\]
to find a bound for the error in the approximation in (a).

**Solution:** (This is a little bit more complicated than I had originally intended...) Here, since we’ve used the Midpoint Riemann Sum, \( N \) represents an upper bound on the second derivative \( f(t) = e^{1/t} \) on the interval \([1, 2]\). Now, if \( f(t) = e^{1/t} \), then \( f'(t) = e^{1/t} \cdot \frac{1}{t^2} \), and so \( f''(t) = e^{1/t} \cdot \left( \frac{1}{t^4} \right) + e^{1/t} \cdot \frac{2}{t^3} \). So on \([1, 2]\),
\[
|f''(t)| \leq \frac{|e^{1/t}|}{t^4} + 2 \frac{|e^{1/t}|}{t^3}.
\]
Now, since \( g(t) = e^t \) is an increasing function (the \( y \)-values get bigger as \( t \) gets bigger) and on \([1, 2]\),
\[
\frac{1}{2} \leq 1 \leq 1,
\]
we get that \( |e^{1/t}| \leq e < 3 \) for \( t \in [1, 2] \). Therefore,
\[
|f''(t)| \leq \frac{3}{1} + 2 \cdot \frac{3}{1} = 9 =: N.
\]

Therefore,
\[
|error| \leq \frac{9}{2400}.
\]
8. The lifetime $T$ (in hours) of a lightbulb has probability density function

$$f(t) = \begin{cases} 
\frac{1}{100}e^{-t/100} & \text{if } t \geq 0 \\
0 & \text{if } t < 0 
\end{cases}$$

(a) Find the probability that the lightbulb will not burn out within the first 100 hours. The probability that the lightbulb will not burn out within the first 100 hours is just the probability that the lifetime of the lightbulb is greater than 100 hours.

$$P(T > 100) = \int_{100}^{\infty} f(t)dt = \int_{100}^{\infty} \frac{1}{100}e^{-t/100}dt = -\frac{e^{-t/100}}{100} \bigg|_{100}^{\infty} = \frac{1}{e}.$$ 

(b) Find the expected lifetime of the lightbulb.

The expected lifetime of the lightbulb is

$$\mathbb{E}(T) = \int_{-\infty}^{\infty} tf(t)dt = \int_{0}^{\infty} t \cdot \frac{1}{100}e^{-t/100}dt.$$ 

Let $u = t$ and $dv = \frac{1}{100}e^{-t/100}dt$. Then $du = dt$ and $v = -100e^{-t/100}$ and using IBP gives us

$$\mathbb{E}(T) = -te^{-t/100} \bigg|_{0}^{\infty} - \int_{0}^{\infty} -100e^{-t/100}dt$$

$$= 0 - \left(-100e^{-t/100} \bigg|_{0}^{\infty}\right)$$

$$= 100.$$ 

(where L’Hôpital’s Rule was used to evaluate the limit $\lim_{t \to \infty} te^{-t/100}$). Therefore, the expected lifetime of the lightbulb is 100 hours.

9. The distance $X$ (in cm) between a dart’s location on a dartboard and the bullseye (centre of the dartboard) has probability density function

$$f(x) = \begin{cases} 
\frac{3}{4000}(20x - x^2) & \text{if } 0 \leq x \leq 20 \\
0 & \text{otherwise} 
\end{cases}$$

Calculate the standard deviation in $X$, $\sigma(X)$.

First, the expected value is

$$\mathbb{E}(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_{0}^{20} \frac{3}{4000}(20x^2 - x^3)dx$$

$$= \frac{3}{4000} \left( \frac{20}{3}x^3 - \frac{1}{4}x^4 \right) \bigg|_{0}^{20}$$

$$= 10.$$
We also have that

\[ \mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 f(x)dx = \int_0^{20} \frac{3}{4000}(20x^3 - x^4)dx \]

\[ = \frac{3}{4000} \left( 5x^4 - \frac{1}{5}x^5 \right) \bigg|_0^{20} \]

\[ = 120. \]

Therefore the variance is given by

\[ \text{Var}(X) = \mathbb{E}(X^2) = [\mathbb{E}(X)]^2 = 120 - 10^2 = 20, \]

and so the standard deviation is \( \sigma(X) = \sqrt{20}. \)