1. State whether the following are true or false. If true, provide a short justification. If false, provide a counterexample.

(a) If the sequence \((-1)^n a_n\) converges, then the sequence \(a_n\) converges to 0.

Solution: This statement is false. For a counterexample, take \(a_n = \{(−1)^n\}\). Then \((-1)^n a_n = \{(−2)^n\} = \{1\}\), which is a convergent sequence (\(\lim_{n \to \infty} 1 = 1\)), so the hypothesis of the statement is satisfied. However, \(a_n = \{(−1)^n\}\) does not converge to zero (it doesn’t converge at all!), so the conclusion of the statement is not satisfied.

(b) If the sequence \(a_n\) diverges, then \(\lim_{n \to \infty} a_n = \pm \infty\).

Solution: This statement is false. For a counterexample, again we can take \(a_n = \{(−1)^n\}\). This sequence diverges (since the terms of the sequence alternate between 1 and \(-1\), and therefore do not approach a single number), but \(\lim_{n \to \infty} (-1)^n \neq \pm \infty\).

(c) If the series \(\sum_{n=1}^{\infty} a_n\) converges, then \(\sum_{n=1}^{\infty} \cos(a_n)\) converges.

Solution: Again, this statement is false. For a counterexample, take \(a_n = 0\) for all \(n\). Then, \(\sum_{n=1}^{\infty} a_n = \sum_{n=0}^{\infty} 0 = 0\), (i.e. the series converges to 0), but the series \(\sum_{n=1}^{\infty} \cos(a_n) = \sum_{n=1}^{\infty} \cos(0) = \sum_{n=1}^{\infty} 1\), which diverges (by the Divergence Test, since the terms of the series don’t go to 0).

2. The sequence \(a_n\), where \(a_n = \left(1 + \frac{x}{n}\right)^n\), converges for all real numbers \(x\) to some number \(a_x > 0\). Find \(a_x\) (your answer will be in terms of \(x\)).

Solution: We are told that

\[ a_x = \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n, \]

and we wish to find \(a_x\). The limit on the right-hand side is in the indeterminate form \(1^\infty\), so to calculate the limit we take the natural logarithm of both sides:

\[ \ln(a_x) = \ln \left( \lim_{n \to \infty} \left(1 + \frac{x}{n}\right)^n \right) \]
\[ = \lim_{n \to \infty} \ln \left( \left(1 + \frac{x}{n}\right)^n \right) \quad \text{(since } \ln(x) \text{ is a continuous function)} \]
\[ = \lim_{n \to \infty} n \ln \left(1 + \frac{x}{n}\right) \]
Now the limit is in the indeterminate form $\infty \cdot 0$, which we need to rearrange to be in the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ so that we can use L'Hôpital's Rule. We rearrange it to be in the form $\frac{0}{0}$ by writing $n = \left(\frac{1}{x}\right)$. Then

$$\lim_{n \to \infty} n \ln \left(1 + \frac{x}{n}\right) = \lim_{n \to \infty} \frac{\ln \left(1 + \frac{x}{n}\right)}{\frac{1}{n}}$$

$$= \lim_{n \to \infty} \frac{\frac{1}{1 + \frac{x}{n}} \cdot \frac{-x}{n^2}}{\frac{1}{n^2}}$$

$$= \lim_{n \to \infty} \frac{x}{1 + \frac{x}{n}}$$

$$= x$$

Therefore, $\ln(a_x) = x$, which means that $a_x = e^x$.

3. The set $F$ is a fractal constructed as follows.

   (i) Begin with the line segment $I_0 = [0, 1]$.
   (ii) Remove the open middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$, to get $I_1 = [0, \frac{1}{3}] \cup \left[\frac{2}{3}, 1\right]$.
   (iii) Remove the open middle third from every remaining line segment, to get

$$I_2 = \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right] \cup \left[\frac{8}{9}, 1\right]$$.

   (iv) Repeat the process ad infinitum (forever).

A few iterations of this construction are illustrated below.

(a) By considering a suitable series, show that the total length of all of the intervals removed is equal to 1.

Solution: In the first step, when we remove the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$, we remove an interval of length $\frac{1}{3}$. In the second step, when we remove the intervals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$, we remove 2 intervals of length $\frac{1}{9} = \frac{1}{3}$ each. In the third step, we remove 4 = $2^2$ intervals of length $\frac{1}{27} = \frac{1}{3^3}$ each. In each step, we remove the middle third of each interval leftover from the previous step, effectively splitting each of those intervals into two new (smaller) pieces, and since we remove the middle third, we remove a piece whose length is $\frac{1}{3}$ of each of the intervals leftover from the previous step. From this we can see that, in the $k$th step, we'll remove $2^{k-1}$ intervals, each of length $\frac{1}{3^k}$. So, in the $k$th step, we'll remove $\frac{2^{k-1}}{3^k}$ in total. Therefore, the total amount removed (after performing all of the steps) will be given by the series

$$\sum_{k=1}^{\infty} \frac{2^{k-1}}{3^k}.$$
To find the sum of this series, we rewrite the $k$th term as

$$
\frac{2^{k-1}}{3^k} = \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{k-1}
$$

so that we get

$$
\text{total length removed} = \sum_{k=1}^{\infty} \frac{1}{3} \cdot \left(\frac{2}{3}\right)^{k-1}.
$$

Writing it this way, we can recognize this as a convergent geometric series with $a = \frac{1}{3}$ and $r = \frac{2}{3}$ (convergent since $|r| < 1$). So, the sum of the series is just

$$
\frac{a}{1-r} = \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1.
$$

(b) If the total length removed is 1, have we removed all of the points from the interval [0, 1]? Justify your answer.

Solution: No, even though we’ve removed all of the length of the interval we have not removed all of the points. This seems contradictory, but it stems from the fact that an individual point has no length (kind of like a line has no area, and a square has no volume). For example, since we always remove the (open) middle third of each interval, we never remove the endpoints of those intervals (they are never in the middle). So, in particular, the endpoints 0 and 1 are two points that are never removed (and, in fact, there are infinitely many points that are never removed).

4. (Bonus) State whether the following series is convergent or divergent. If it is convergent, find its sum.

$$
\sum_{n=1}^{\infty} \frac{1}{n^3 + 3n^2 + 2n}
$$

Solution: First, since the denominator is larger than $n^3$, and $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges by $p$-series, we expect the given series to also converge (we could show this using the Comparison Test, but since we expect it to converge and are asked to find the sum if it converges, we might as well just go straight to finding the sum).

To find the sum of the series we write

$$
\frac{1}{n^3 + 3n^2 + 2n} = \frac{1}{n(n+1)(n+2)} = \frac{A}{n} + \frac{B}{n+1} + \frac{C}{n+2}.
$$

Multiplying everything by $n(n+1)(n+2)$ we have

$$
1 = A(n+1)(n+2) + Bn(n+2) + Cn(n+1).
$$

Plugging in $n = 0$ we see that $A = \frac{1}{2}$. Plugging in $n = -1$, we see that $B = -1$. Plugging in $n = -2$, we see that $C = \frac{1}{2}$. Therefore,

$$
\sum_{n=1}^{\infty} \frac{1}{n^3 + 3n^2 + 2n} = \sum_{n=1}^{\infty} \left(\frac{1}{2n} + \frac{-1}{n+1} + \frac{1}{2(n+2)}\right)
$$

So, the $n$th partial sum (the sum of the first $n$ terms) of this series is

$$
S_n = \left(\frac{1}{2} + \frac{-1}{2} + \frac{1}{3}\right) + \left(\frac{1}{2} + \frac{-1}{3} + \frac{1}{4}\right) + \left(\frac{1}{3} + \frac{-1}{4} + \frac{1}{5}\right) + \ldots
$$

$$
\ldots + \left(\frac{1}{n-1} + \frac{-1}{n} + \frac{1}{n+1}\right) + \left(\frac{1}{n} + \frac{-1}{n+1} + \frac{1}{n+2}\right).
$$
Looking carefully at this, we see that the rightmost term in each bracket will cancel with the middle term of the following bracket and the leftmost term of the bracket two after. That is

\[ S_n = \left( \frac{1}{2} + \frac{-1}{2} + \frac{1}{3} \right) + \left( \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \right) + \left( \frac{1}{3} + \frac{-1}{4} + \frac{1}{5} \right) + \ldots \]

\[ \ldots + \left( \frac{1}{2n - 1} + \frac{-1}{n} + \frac{1}{n + 1} \right) + \left( \frac{1}{n} + \frac{-1}{n + 1} + \frac{1}{n + 2} \right). \]

So, after doing all of the cancellation, we’re left with

\[ S_n = \frac{1}{2} + \frac{-1}{2} + \frac{1}{2} + \frac{1}{n + 1} + \frac{-1}{n + 1} + \frac{1}{n + 2} = \frac{1}{4} + \frac{-\frac{1}{2}}{n + 1} + \frac{\frac{1}{2}}{n + 2}. \]

Therefore the sum of the series is

\[ \text{sum} = \lim_{n \to \infty} S_n = \lim_{n \to \infty} \frac{1}{4} + \frac{-\frac{1}{2}}{n + 1} + \frac{\frac{1}{2}}{n + 2} = \frac{1}{4}. \]