Short Answer Questions

1. Evaluate the following integrals or state that they diverge.
   
   (a) \( \int \sin^3(x) \, dx \) \hspace{1cm} (Answer: \( \frac{1}{3} \cos^3(x) - \cos(x) + C \))

   (b) \( \int \frac{t}{\sqrt{4 + t^2}} \, dt \) \hspace{1cm} (Answer: \( \sqrt{4 + t^2} + C \))

   (c) \( \int_0^\infty \frac{2}{\sqrt{7 - x^2}} \, dx \) \hspace{1cm} (Answer: \( \frac{\pi}{12} + \frac{7\sqrt{3}}{8} \))

   (d) \( \int \frac{1}{t(t-1)^2} \, dt \) \hspace{1cm} (Answer: \( \ln |t| - \ln |t-1| - \frac{1}{t-1} + C \))

   (e) \( \int_2^\infty \frac{1}{t \ln(t)} \, dt \) \hspace{1cm} (Answer: The integral diverges.)

2. Find the area below \( y = x \ln(x) \) and above \( y = \cos^4(x) \) between \( x = \pi \) and \( x = 2\pi \).
   (Answer: \( 2\pi^2 \ln(2\pi) - \frac{1}{2} \pi^2 \ln(\pi) - \frac{3}{4} \pi^2 - \frac{3}{8} \pi \))

3. Find the expected value of the probability density function
   \[
   f(x) = \begin{cases} 
   4e^{-4x} & \text{if } x \geq 0 \\
   0 & \text{if } x < 0 
   \end{cases}
   \]
   (Answer: \( \frac{1}{4} \))

4. Find the area beneath the curve \( y = \frac{1}{t^2 + 6t + 8} \) from \( t = 0 \) to \( t = 1 \).
   (Answer: \( \frac{1}{2} \ln \left( \frac{6}{5} \right) \))

5. Use a left Riemann sum with \( n = 40 \) to approximate the area below the graph of \( f(x) = \sin(e^{2x}) \) from \( x = 2 \) to \( x = 6 \).
   (Answer: \( \int_2^6 \sin(e^{2x}) \, dx \approx \sum_{i=1}^{40} \sin \left( e^{2 + \frac{1}{18} (i-1)} \right) \cdot \frac{1}{18} \))

6. Estimate the error in the approximation in (5).
   (Answer: \( |\text{error}| \leq \frac{e^{12}}{18} \))
7. If \( G(x) = \int_0^x \cos(\cos(\cos(t))) dt \), find \( G'(\frac{\pi}{2}) \). \( \text{Answer: } G'(\frac{\pi}{2}) = \cos(1) \)

Note: There was originally a typo in this problem – it read \( G(x) = x \int_0^x \cos(\cos(\cos(t))) dt \), but there wasn’t supposed to be an \( x \) multiplying the integral. The answer is for the corrected version.

8. Of the vectors \( (1, 2, 0), (-1, 0, 3), \) and \( (4, -2, 3) \), which two are perpendicular? \( \text{Answer: } (1, 2, 0) \) and \( (4, -2, 3) \)

9. If \( f(x, y) = 1 + 2x\sqrt{y} \), find \( \frac{\partial^2 f}{\partial x \partial y} \). \( \text{Answer: } \frac{1}{\sqrt{y}} \)

10. Find the equation of the plane tangent to \( f(x, y) = y \ln(x) \) at the point \( (1, 4, 0) \). \( \text{Answer: } z = 4(x - 1) \)

11. If \( f(x, y) = \sin(2x + 5y) \), find \( \nabla f(x, y) \). \( \text{Answer: } (2 \cos(2x + 5y), 5 \cos(2x + 5y)) \)

12. Find the rate of change \( f(x, y) = \cos(xy) + 2xe^y \) in the direction \( (1, 2) \) at the point \( (1, 0) \). \( \text{Answer: } \frac{6}{\sqrt{5}} \)

13. Find the maximum rate of change of \( f(x, y) = y^2 x \) at the point \( (2, 4) \) and the direction in which it occurs. \( \text{Answer: } \sqrt{32}, (-4, 4) \)

14. For each sequence/series, find the limit/sum or state that it diverges.
   
   (a) \( \left\{ 1 + \frac{(-1)^n}{\sqrt{n}} \right\} \). \( \text{Answer: } 1 \)

   (b) \( \left\{ \frac{n^2 + 5}{\sqrt{3n^2 + 13}} \right\} \). \( \text{Answer: } \frac{1}{\sqrt{3}} \)

   (c) \( \{(-1)^n \frac{n+1}{n}\} \). \( \text{Answer: } \text{Diverges} \)

   (d) \( \sum_{k=1}^{\infty} \frac{4}{k^2 + 2k} \). \( \text{Answer: } 3 \)

   (e) \( \sum_{k=1}^{\infty} e^{\frac{4k-1}{52k}} \). \( \text{Answer: } \frac{e^2}{11} \)

   (f) \( \sum_{k=0}^{\infty} \frac{2^k}{k!} \). \( \text{Answer: } e^2 \)
15. For each of the following, determine whether the series converges absolutely, conditionally or diverges.

(a) $\sum_{k=7}^{\infty} \frac{k}{k^\pi - 2}$.  \hspace{1cm} \text{(Answer: Converges absolutely)}

(b) $\sum_{k=1}^{\infty} \frac{k}{2^k}$.  \hspace{1cm} \text{(Answer: Converges absolutely)}

(c) $\sum_{k=4}^{\infty} \frac{e^{7/k}}{k^2}$.  \hspace{1cm} \text{(Answer: Converges absolutely)}

(d) $\sum_{k=10}^{\infty} (-1)^k e^{-k}$.  \hspace{1cm} \text{(Answer: Converges absolutely)}

(e) $\sum_{k=0}^{\infty} \cos(\pi k)$.  \hspace{1cm} \text{(Answer: Diverges)}

(f) $\sum_{k=17}^{\infty} \frac{2^k \cos(k)}{k!}$.  \hspace{1cm} \text{(Answer: Converges absolutely)}

(g) $\sum_{k=2}^{\infty} \frac{(-1)^k}{k \ln(k)}$.  \hspace{1cm} \text{(Answer: Converges conditionally)}

16. Find power series representations for the following functions.

(a) $\frac{x}{4 + x^2}$  \hspace{1cm} \text{(Answer: $\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{4^{k+1}}$)}

(b) $\int x \cos(3x^4)dx$  \hspace{1cm} \text{(Answer: $\sum_{k=0}^{\infty} \frac{(-1)^k 3^{2k} x^{8k+2}}{(2k)! (8k + 2)} + C$)}

(c) $\frac{x^2 - 1}{x^2}$  \hspace{1cm} \text{(Answer: $\sum_{k=1}^{\infty} \frac{x^{2k-2}}{k!}$)}

17. Find the interval of convergence of the following power series.

(a) $\sum_{k=0}^{\infty} (-1)^k \frac{10^k (x + 2)^k}{k^2}$  \hspace{1cm} \text{(Answer: $[-\frac{21}{10}, -\frac{19}{10}]$)}

(b) $\sum_{k=0}^{\infty} \frac{k^2 x^k}{2 \cdot 4 \cdot 6 \cdots (2k)}$  \hspace{1cm} \text{(Answer: $(-\infty, \infty)$)}
18. Find the sum of the series \( \sum_{k=1}^{\infty} \frac{3^k}{5^k k!} \). (Answer: \( e^{3/5} - 1 \))

19. Use a series to evaluate the limit.

\[
\lim_{x \to 0} \frac{\sin(x) - x + \frac{1}{6} x^3}{x^5}
\]

(Answer: \( \frac{1}{3!} \))

**Long Answer Questions**

1. The Gompertz equation

\[
\frac{dP}{dt} = -27 \ln \left( \frac{P}{120} \right) P,
\]

is a differential equation used to model limited populations of phytoplankton.

(a) If the initial population of phytoplankton in a pond is \( P(0) = 60 \), and if the population of the phytoplankton satisfies the Gompertz equation, find population as a function of \( t \) (i.e. find \( P(t) \)). (Answer: \( P(t) = 120 e^{\ln(\frac{1}{2}) e^{-27t}} \))

(b) What will happen to the population of phytoplankton in the long run? (Answer: The population will approach 120.)

2. Find the critical points of the function \( f(x,y) = e^y (y^2 - x^2) \) and classify them as being local maxima, local minima or saddle points. (Answer: Saddle point at \((0,0)\), local max at \((0,-2)\))

3. Find the point on the plane \( x - y + z = 4 \) that is closest to the point \((1,2,3)\). (Answer: \((2,2,4)\))

4. Use the method of Lagrange multipliers to find the global maximum and the global minimum of the function \( f(x,y) = x^2 y \) subject to the constraint \( x^2 + 2y^2 = 6 \). (Answer: Global maxima of 4 at \((2,1)\), \((-2,1)\), global minima of \(-4\) at \((2,-1), (-2,-1)\).)

5. Find the global maximum and global minimum values of the function \( f(x,y) = e^{xy + x} \) subject to the constraint \( x^2 + y^2 \leq 4 \). (Answer: Global maxima of about 33.8 at \( \left( \frac{-1+\sqrt{33}}{4}, \sqrt{\frac{30+2\sqrt{33}}{16}} \right) \), global minimum of about 0.03 at \( \left( \frac{-1+\sqrt{33}}{4}, -\sqrt{\frac{30+2\sqrt{33}}{16}} \right) \))

**Note:** You would not be expect to figure out which point corresponds to the max and which corresponds to the min without a calculator for this problem.

6. A certain ball has the property that each time it falls from a height \( h \) onto a hard, level surface, it rebounds to a height \( rh \), where \( 0 < r < 1 \) is a constant. Suppose the ball is dropped from an initial height of 5m and that \( r = \frac{3}{4} \).
(a) Assuming the ball continues to bounce indefinitely, find the total distance that it travels. \((\textbf{Answer: } 35\text{m})\)

(b) Suppose that each time the ball strikes the surface with velocity \(v\), it rebounds with velocity \(-kv\), where \(0 < k < 1\). How long will it take for the ball to come to rest? \((\textbf{Answer: } \text{It will never come to rest (an infinite amount of time).})\)

7. Find the Taylor series for \(f(x) = x^{-\frac{3}{2}}\) centred at \(x = 1\). \((\textbf{Answer: } \sum_{k=0}^{\infty} \frac{(-1)^k(1)(3)\cdots(2k+1)}{2^k \cdot k!}(x-1)^k)\)