1. Let $S$ be a countable set and $E$ be any infinite subset of $S$. Show that $E$ is countable.

   Ans: Since $S$ is countable, there exists a bijection $f: \mathbb{N} \rightarrow S$, where $\mathbb{N}$ is the set of natural numbers $\{1, 2, 3, \ldots\}$. Let $E$ be any infinite subset of $S$ and let $n_1$ be the smallest integer such that $f(n_1) \in E$. Such an $n_1$ exists by the Well Ordering property. Let $n_2$ be the smallest positive integer such that $n_2 > n_1$ and $f(n_2) \in E$. Inductively define $n_k$ to be the smallest integer (positive) greater than $n_{k-1}$ such that $f(n_k) \in E$. This process continues indefinitely since $E$ is infinite. Consider the function $g: \mathbb{N} \rightarrow E$, by $g(1) = f(n_1), g(2) = f(n_2), \ldots$. Clearly the map $g$ is injective as $f$ is injective. To show that $g$ is surjective, let $e \in E$. Since $E \subseteq S$, and $f$ is surjective, we see that $E = \{f(n_1), f(n_2), f(n_3), \ldots\}$. Hence $g(n) = f(n)$ = $e$ and $g$ is a bijection. Hence $E$ is countable.

2. Show that the product of any three consecutive integers is divisible by 3.

   Ans: Let $a, a+1, a+2$ be any three consecutive integers. We may clearly assume $a \neq 0$. Then one of the three integers has to be divisible by 3, and therefore the same holds for the product.

3. For a prime number $p > 3$, show that the triplet $p, p+2, p+4$ are not all primes.
Ans: Since \( p > 3 \) is a prime, \( p \times 3 \). Thus if we divide \( p \) by 3, then \( p = 3q + r \), with \( 0 \leq r < 3 \).

9) If \( r = 0 \), then \( p \) is divisible by 3 \( \Rightarrow \) \( p \) is prime.

9) If \( r = 1 \), then \( p = 3k + 1 \) \( \Rightarrow \) \( p + 2 \) is divisible by 3.

So \( (p + 2) \) is not a prime.

9) If \( r = 2 \), then \( p = 3k + 2 \) \( \Rightarrow \) \( p + 4 = 3k + 6 \) and \( 3 \mid (p + 4) \), so \( (p + 4) \) is not a prime.

3) Show that there are infinitely many primes of the form \( 30n + 7 \), \( n \in \mathbb{N} \).

Ans: The gcd \((7, 30) = 1\). The set \( \{30n + 7\} \) is in arithmetic progression and we can apply Dirichlet's Theorem. Hence there are infinitely many primes of this form.

4) Show that the 5,987,091st prime is likely to have 7 digits; given that \( x/\ln x \approx 72,382.4 \) for \( x = 10^6 \) and \( x/\ln x \approx 620,420.7 \) for \( x = 10^7 \).

Ans: The prime number theorem tells us that \( x/\ln x \) is a good approximation for \( \pi(x) \) as \( x \to \infty \). Take \( x = 10^6 \), and \( x = 10^7 \), then \( \pi(x) \) counts the number of primes \( \leq 10^6 \). Since the \( k \)th prime \( p \) for \( k = 5,987,091 \), has to lie between \( \pi(10^6) \) and \( \pi(10^7) \), given the values for \( x/\ln x \) when \( x = 10^6 \) and \( 10^7 \), we see that it has 7 digits.